

THIN POSITION FOR KNOTS, LINKS, AND GRAPHS IN 3-MANIFOLDS

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ABSTRACT. We define a new notion of thin position for a graph in a 3-manifold which combines the ideas of thin position for manifolds first originated by Scharlemann and Thompson with the ideas of thin position for knots first originated by Gabai. This thin position has the property that connect summing annuli and pairs-of-pants show up as thin levels. In a forthcoming paper, this new thin position allows us to define two new families of invariants of knots, links, and graphs in 3-manifolds. The invariants in one family are similar to bridge number and the invariants in the other family are similar to Gabai's width for knots in the 3-sphere. The invariants in both families detect the unknot and are additive under connected sum and trivalent vertex sum.

1. INTRODUCTION

In [3], Gabai introduced *width* as an extremely useful knot invariant. Width is a certain function (whose exact definition isn't needed for our purposes) from the set of height functions on a knot in S^3 to the natural numbers. It becomes an invariant after minimizing over all possible height functions. A particular height function of a knot is *thin* if it realizes the minimum width. Thin embeddings produce very useful topological information about the knot (see, for example, Gabai's proof of Property R in [3]; Gordon and Luecke's solution to the knot complement problem [21]; and Thompson's proof [23] that small knots have thin position equal to bridge position.) Scharlemann and Thompson extended Gabai's width for knots to a width for graphs in S^3 [14] and gave a new proof of Waldhausen's classification of Heegaard splittings of S^3 [20]. They also applied a similar idea to handle structures of 3-manifolds, producing an invariant of 3-manifolds also called width [13]. A handle decomposition which attains the width is said to be *thin*. Thin handle decompositions for 3-manifolds have been very useful for understanding the structure of Heegaard splittings of 3-manifolds.

There have been a number of attempts (eg. [1, 5, 6, 18, 22]) to define width of knots (and later for tangles and graphs) in a 3-manifold by using various generalizations of the Scharlemann-Thompson constructions. These definitions have been used in various ways, however they have never been as useful as Scharlemann and Thompson's thin position for 3-manifolds. For instance, although Scharlemann and Thompson's thin position has the property that all thin surfaces in a thin handle decomposition for a closed 3-manifold are essential surfaces the same is not true for the thin positions applied to knot and graph complements. (The papers [1] and [18] are exceptions. The former, however, applies only to links in closed 3-manifolds and in the latter there are a number of technical requirements which limit its utility.)

In this paper, we define an *oriented v.p.-bridge surface* as a certain type surface \mathcal{H} in a 3-manifold M transverse to a graph $T \subset M$. The components of \mathcal{H} are partitioned into *thick surfaces* \mathcal{H}^+ and *thin surfaces* \mathcal{H}^- . We exhibit a collection of thinning moves which give rise to a partial order, denoted \rightarrow , on the set of *reduced, oriented, multiple v.p.-bridge surfaces* (terms to be defined later) for a (3-manifold, graph) pair (M, T) . These thinning moves include the usual kinds of destabilization and untelescoping moves, known to experts, but we include several new ones, corresponding

to the situation when portions of the graph T are cores of compressionbodies in a generalized Heegaard splitting of M (in the sense of [13].) More significantly, we also allow untelescoping using various generalizations of compressing discs. Throughout the paper, we show how these generalized compressing discs arise naturally when considering bridge surfaces for (3-manifold, graph) pairs. If \mathcal{H} and \mathcal{K} are oriented bridge surfaces, we say that $\mathcal{H} \rightarrow \mathcal{K}$ if certain kinds of carefully constructed sequences of thinning moves produce \mathcal{K} from \mathcal{H} . If no such sequence can be applied to \mathcal{H} then we say that \mathcal{H} is *locally thin*. If the reader allows us to defer some more definitions until later, we can state our results as:

Main Theorem. *Let M be a compact, orientable 3-manifold and $T \subset M$ a properly embedded graph such that no vertex has valence two and no component of ∂M is a sphere intersecting T two or fewer times. Assume also that every sphere in M separates and that no sphere in M intersects T exactly once transversally. Then \rightarrow is a partial order on the set of reduced elements of $\overrightarrow{vp\mathbb{H}}(M, T)$. Furthermore, if $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ is reduced, then there is a locally thin reduced $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$ such that $\mathcal{H} \rightarrow \mathcal{K}$. Additionally, if \mathcal{H} is locally thin then the following hold:*

- (1) *Each component of \mathcal{H}^+ is sc-strongly irreducible in the complement of the thin surfaces.*
- (2) *No component of $(M, T) \setminus \mathcal{H}$ is a trivial product compressionbody between a thin surface and a thick surface.*
- (3) *Every component of \mathcal{H}^- is c-essential in the exterior of T*
- (4) *If there is a 2-sphere in M intersecting T three or fewer times and which is essential in the exterior of T , then some component of \mathcal{H}^- is such a sphere.*

The properties of locally thin surfaces are proved in part by using sweep-out arguments. (See Theorems 8.6 and 9.2.) The existence of a locally thin \mathcal{K} with $\mathcal{H} \rightarrow \mathcal{K}$ is proved using a new (rather complicated) complexity which decreases under thinning sequences. (See Theorem 7.4.) Although this complexity functions much as Gabai's or Scharlemann-Thompson's widths do, we view it as being more like the complexities used to guarantee that hierarchies of 3-manifolds terminate. As we observe, there are modifications of the complexity, which are equally effective at guaranteeing the thinning sequences cannot be perpetually applied. In the sequel [19] we will show how powerful these locally thin positions for (3-manifold, graph) pairs are. In that paper, we construct two families of non-negative half-integer invariants of (3-manifold, graph) pairs. The invariants of one family are similar to the bridge number and tunnel number of a knot. The invariants of the other family are very similar to Gabai's width for knots in S^3 . We prove that these invariants (under minor hypotheses) are additive for both connect sum and trivalent vertex sum and detect the unknot.

In Section 2 and Section 3 we establish our notation and important definitions including the definition of a multiple v.p.-bridge surface. We describe our simplifying moves in Sections 4 and 5. In Section 6, we define a complexity for oriented multiple v.p.-bridge surfaces and show it decreases under our simplifying moves. Section 7 uses the simplifying moves to define a partial order \rightarrow on the set $\overrightarrow{vp\mathbb{H}}(M, T)$ of oriented multiple v.p.-bridge surfaces for (M, T) . The main theorem, Theorem 7.4, shows that given a reduced $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ there is a least element $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$ with respect to the partial order \rightarrow such that $\mathcal{H} \rightarrow \mathcal{K}$. The least elements are called "locally thin." In Section 8, we study the important properties of locally thin multiple v.p.-bridge surfaces. Theorem 8.6 lists a number of these properties, one of which is that each component of \mathcal{H}^- is essential in the exterior of T . Section 9 sets us up for working with connected sums in [19] by showing that if there is a sphere in M , transversely intersecting T in three or fewer points, and which is essential in the exterior of T , then there is such a sphere that is a thin level for any locally thin multiple v.p.-bridge surface.

Acknowledgements. Some of this paper is similar in spirit to [18], but here we operate under much weaker hypotheses and obtain much stronger results. We have endeavored to make this paper as independent as possible from [18]. As in [18], we have been heavily influenced by Hayashi and Shimokawa’s work in [5]. Throughout we assume some familiarity with the theory of Heegaard splittings, as in [11]. We thank Ryan Blair, Marion Campisi, Jesse Johnson, Alex Zupan, and the attendees at the 2014 “Thin Manifold” conference for helpful conversations.

2. DEFINITIONS AND NOTATION

We let $I = [-1, 1] \subset \mathbb{R}$, D^2 be the closed unit disc in \mathbb{R}^2 , and B^3 be the closed unit ball in \mathbb{R}^3 . For a topological space X , we let $|X|$ denote the number of components of X . All surfaces and 3-manifolds we consider will be orientable, smooth or PL, and (most of the time) compact. If S is a surface, then $g(S)$ is its genus. We will use the common convention that the genus of a disconnected surface is the sum of the genera of its connected components.

A **(3-manifold, graph) pair** (M, T) (or simply just a **pair**) consists of a compact, orientable 3-manifold (possibly with boundary) M and a properly embedded graph $T \subset M$. We do not require T to have vertices so T can be empty or a knot or link. Since T is properly embedded in M all valence 1 vertices lie on ∂M . We call the valence 1-vertices of T the **boundary vertices** or **leaves** of T and all other vertices the **interior vertices** of T . We require that no vertex of T have valence 0 or 2, but we allow a graph to be empty.

For any subset X of M , let $\eta(X)$ be an open regular neighborhood of X in M and $\overline{\eta(X)}$ be its closure. If S is a (orientable, by convention) surface properly embedded in M and transverse to T , we write $S \subset (M, T)$. If $S \subset (M, T)$, we abuse notation slightly and write

$$(M, T) \setminus S = (M \setminus S, T \setminus S) = (M \setminus \eta(S), T \setminus \eta(S)).$$

We also write $S \setminus T$ for $S \setminus \eta(T)$. Observe that $\partial(M \setminus T)$ is the union of $(\partial M) \setminus T$ with $\partial\overline{\eta(T)}$. A surface $S \subset (M, T)$ is **∂ -parallel** if $S \setminus T$ is isotopic relative to its boundary into $\partial(M \setminus T)$. We say that $S \subset (M, T)$ is **essential** if $S \setminus T$ is incompressible in $M \setminus T$, not ∂ -parallel, and not a 2-sphere bounding a 3-ball in $M \setminus T$. We say that the graph $T \subset M$ is **irreducible** if whenever $S \subset (M, T)$ is a 2-sphere we have $|S \cap T| \neq 1$. The pair (M, T) is **irreducible** if T is irreducible and if the 3-manifold $M \setminus T$ is irreducible (i.e. does not contain an essential sphere.)

We will need notation for a few especially simple (3-manifold, graph) pairs. The pair (B^3, arc) will refer to any pair homeomorphic to the pair (B^3, T) with T an arc properly isotopic into ∂B^3 . The pair $(S^1 \times D^2, \text{core loop})$ will refer to any pair homeomorphic to the pair $(S^1 \times D^2, T)$ where T is the product of S^1 with the center of D^2 .

Finally, we will often convert vertices of T into boundary components of M and vice versa. More precisely, if V is the union of all the interior vertices of T , we say that $(\overset{\circ}{M}, \overset{\circ}{T}) = (M \setminus \eta(V), T \setminus \eta(V))$ is obtained by **drilling out the vertices of T** . Similarly, we will sometimes refer to **drilling out** certain edges of T ; i.e. removing an open regular neighborhood of those edges and incident vertices from both M and T .

2.1. Compressing discs of various kinds. We will be concerned with several types of discs which generalize the classical definition of a compressing disc for a surface in a 3-manifold.

Definition 2.1. Suppose that $S \subset (M, T)$ is a surface. Suppose that D is an embedded disc in M such that the following hold:

- (1) $\partial D \subset (S \setminus T)$, the interior of D is disjoint from S , and D is transverse to T .

- (2) $|D \cap T| \leq 1$
- (3) D is not properly isotopic into $S \setminus T$ in $M \setminus T$ via an isotopy which keeps the interior of D disjoint from S until the final moment. Equivalently, there is no disc $E \subset S$ such that $\partial E = \partial D$ and $E \cup D$ bounds either a 3-ball in M disjoint from T or a 3-ball in M whose intersection with T consists entirely of a single unknotted arc with one endpoint in E and one endpoint in D .

Then D is an **sc-disc**. If $|D \cap T| = 0$ and ∂D does not bound a disc in $S \setminus T$, then D is a **compressing disc**. If $|D \cap T| = 0$ and ∂D does bound a disc in $S \setminus T$, then D is a **semi-compressing disc**. If $|D \cap T| = 1$ and ∂D does not bound an unpunctured disc or a once-punctured disc in $S \setminus T$, then D is a **cut disc**. If $|D \cap T| = 1$ and ∂D does bound an unpunctured disc or a once-punctured disc in $S \setminus T$, then D is a **semi-cut disc**. A **c-disc** is a compressing disc or cut disc. The surface $S \subset (M, T)$ is **c-incompressible** if S does not have a c-disc; it is **c-essential** if it is essential and c-incompressible.

Remark 2.2. Semi-cut discs arise naturally when T has an edge containing a local knot, as in Figure 1. Semi-compressing discs arise naturally when a 3-manifold has multiple 2-sphere boundary components. Semi-compressing discs, on the other hand, occur in part because even though a 3-manifold M may be irreducible, there is no guarantee that a given 3-dimensional submanifold is also irreducible.

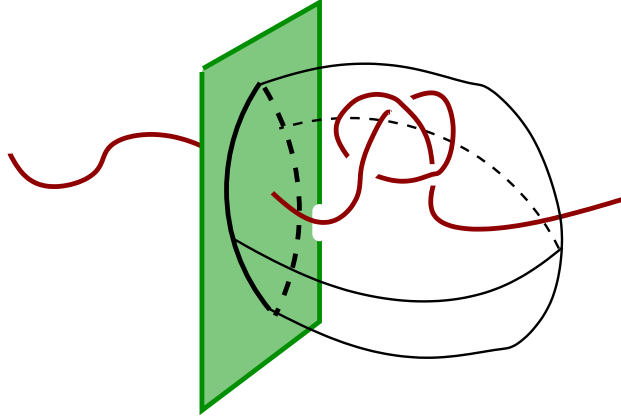


FIGURE 1. Except in very particular situations, the black disc is a semicut disc for the green surface.

3. COMPRESSIONBODIES AND MULTIPLE V.P.-BRIDGE SURFACES

3.1. Compressionbodies. In this section we generalize the idea of a compressionbody to our context.

Definition 3.1. Suppose that H is a closed, connected, orientable surface. We say that $(H \times I, T)$ is a **trivial product compressionbody** or a **product region** if T is isotopic to the union of vertical arcs, and we let $\partial_{\pm}(H \times I) = H \times \{\pm 1\}$. If B is a 3-ball and if $T \subset B$ is a (possibly empty) connected, properly embedded, ∂ -parallel tree, having at most one interior vertex T , then we say that (B, T) is a **trivial ball compressionbody**. We let $\partial_+ B = \partial B$ and $\partial_- B = \emptyset$. A **trivial compressionbody** is either a trivial product compressionbody or a trivial ball compressionbody. Figure 2 shows both types of trivial compressionbodies.

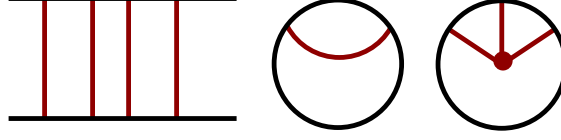


FIGURE 2. On the left is a trivial product compressionbody; in the center is a trivial ball compressionbody with T an arc; on the right is a trivial ball compressionbody with T a tree having a single interior vertex..

A pair (C, T) is a **v.p.-compressionbody** if there is some component denoted $\partial_+ C$ of ∂C and a collection of pairwise disjoint sc-discs $\mathcal{D} \subset (C, T)$ for $\partial_+ C$ such that the result of ∂ -reducing (C, T) using \mathcal{D} is a union of trivial compressionbodies. Observe that trivial compressionbodies are v.p.-compressionbodies as we may take $\mathcal{D} = \emptyset$. Figure 3 shows two different v.p.-compressionbodies. We will usually represent v.p.-compressionbodies more schematically as in Figure 4.

The set $\partial C \setminus \partial_+ C$ is denoted by $\partial_- C$. If no two discs of \mathcal{D} are parallel in $C \setminus T$ then \mathcal{D} is a **complete collection of discs** for (C, T) . An edge of T which is disjoint from $\partial_+ C$ (and so has endpoints on $\partial_- C$ and the vertices of T) is a **ghost arc**. An edge of T with one endpoint in $\partial_+ C$ and one endpoint in $\partial_- C$ is a **vertical arc**. A component of T which is an arc having both endpoints on $\partial_+ C$ is a **bridge arc**. A component of T which is homeomorphic to a circle and is disjoint from ∂C is called a **core loop**. C is a **compressionbody** if (C, \emptyset) is a v.p.-compressionbody. A compressionbody C is a **handlebody** if $\partial_- C = \emptyset$. A **bridge disc** for $\partial_+ C$ in C is an embedded disc in C with boundary the union of two arcs α and β such that $\alpha \subset \partial_+ C$ joins distinct points of $\partial_+ C \cap T$ and β is a bridge arc of T .

Remark 3.2. Suppose that (C, T) is a v.p.-compressionbody and that $(\mathring{C}, \mathring{T})$ is the result of drilling out the vertices of T . Considering the components of $\partial \mathring{C} \setminus \partial C$ as components of $\partial_- C$, we see that $(\mathring{C}, \mathring{T})$ is also a v.p.-compressionbody having the same complete collection of discs as (C, T) . Furthermore, every component of \mathring{T} is a vertical arc, ghost arc, bridge arc, or core loop. The “v.p.” stands for “vertex-punctured” as this notion of compressionbody is a generalization of the compressionbodies used in [18]: the v.p.-compressionbody $(\mathring{C}, \mathring{T})$ satisfies [18, Definition 2.1] with $\Gamma = \mathring{T}$ (in the notation of that paper.) The notation “v.p.” will also be helpful as a reminder that the first step in calculating many of the various quantities we consider is to drill out the vertices of T and treat them as boundary components of M .

The next two lemmas have a straightforward proofs using innermost disc/outermost arc arguments, but some care is needed and so we include them.

Lemma 3.3. *Suppose that (C, T) is a v.p.-compressionbody such that no spherical component of $\partial_- C$ intersects T exactly once. Suppose $P \subset (C, T)$ is a closed surface. If D is an sc-disc for P , then if $|D \cap T| = 1$ either ∂D is essential on $P \setminus T$ or ∂D bounds a disc in P intersecting T exactly once. Furthermore, if P is a sphere, then either P bounds a 3-ball in $C \setminus T$, bounds a 3-ball in C intersecting T in a single unknotted arc, or after some sc-compressions becomes the union of spheres, each parallel to a component of $\partial_- C$ and each intersecting T exactly the same number of times as P .*

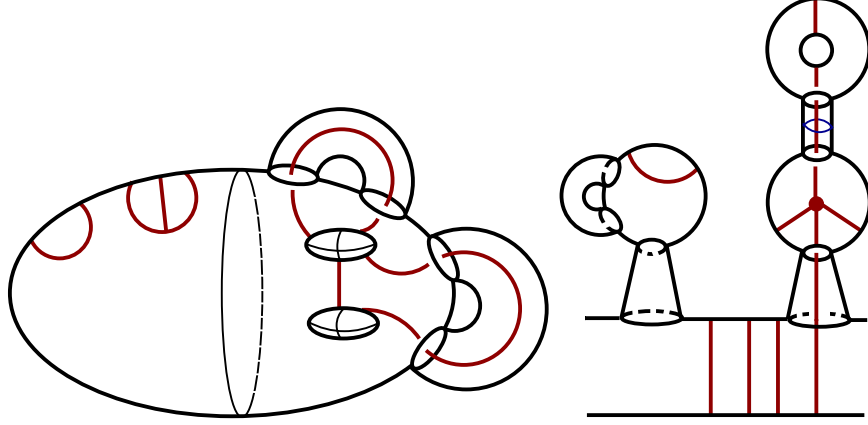


FIGURE 3. On the left is an example of a v.p. compressionbody (C, T) with $\partial_- C$ the union of spheres. On the right, is an example of a v.p. compressionbody (C, T) with $\partial_- C$ the union of two connected surfaces, one of which is a sphere twice-punctured by T .

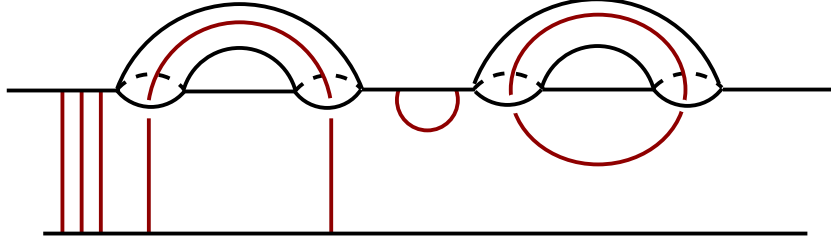


FIGURE 4. A v.p.-compressionbody (C, T) . From left to right we have three vertical arcs, one ghost arc, one bridge arc, and one core loop in T .

Proof. Suppose first that there is an sc-disc D for P such that ∂D bounds an unpunctured disc $E \subset P$ but $|D \cap T| = 1$. Then $E \cup D$ is a sphere in (C, T) intersecting T exactly once. Every sphere in C must separate C . Let $W \subset C \setminus (E \cup D)$ be the component disjoint from $\partial_+ C$. The fundamental group of every component of $\partial_- C$ injects into the fundamental group of C and every curve on every component of $\partial_- C$ is homotopic into $\partial_+ C$. Thus, any component of $\partial_- C$ contained in W must be a sphere. Drilling out the vertices of T along with all edges of T disjoint from $\partial_+ C$ creates a new v.p.-compressionbody (C', T') . As before, any essential curve in $\partial_- C'$ is non-null-homotopic in C and is homotopic into $\partial_+ C$. Thus, $\partial_- C \cap W$ can contain no essential curves. There is an edge $e \subset (T \cap W)$ with an endpoint in D . Beginning with e , traverse a path across edges of $T \cap W$ and components of $\partial_- C \cap W$ (each necessarily a sphere) so that no edge of $T \cap W$ is traversed twice. The path terminates when it reaches a component of $\partial_- C \cap W$ which is a once-punctured sphere, contrary to our hypotheses. Thus, no such sc-disc D can exist.

Suppose, now that P is a sphere. Let Δ be a complete collection of discs for (C, T) chosen so as to minimize $|\Delta \cap P|$ up to isotopy of Δ . If $P \cap \Delta = \emptyset$, then either P bounds a 3-ball in $C \setminus T$, bounds a 3-ball intersecting T in a single unknotted arc, or is ∂ -parallel in (C, T) to a component of $\partial_- C$, and we are done. Otherwise, $\Delta \cap P$ consists of circles. Since we have minimized $|\Delta \cap P|$ up to isotopy, each circle of $\Delta \cap P$ which is innermost on Δ bounds an sc-disc $D \subset \Delta$ for P . By the previous paragraph, compressing P using D creates two spheres in (C, T) each with the same number of punctures as P . Since D was an sc-disc for P , neither of those components is a 2-sphere

bounding a 3-ball disjoint from T or intersecting T in a single unknotted arc. The result follows by repeatedly performing such compressions until P becomes disjoint from Δ . \square

Lemma 3.4. *Suppose that (C, T) is a v.p.-compressionbody such that no component of $\partial_- C$ is a 2-sphere intersecting T exactly once. The following are true:*

- (1) (C, T) is a trivial compressionbody if and only if there are no sc-discs for $\partial_+ C$.
- (2) There are no c-discs for $\partial_- C$.
- (3) If D is an sc-disc for $\partial_+ C$, then reducing (C, T) using D is the union of v.p.-compressionbodies. Furthermore, there is a complete collection of discs for (C, T) containing D .

Proof. Proof of (1): From the definition of v.p.-compressionbody, if there is no sc-disc for $\partial_+ C$, then (C, T) is a trivial compressionbody. The converse requires a little more work, but follows easily from standard results in 3-dimensional topology.

Proof of (2): This is similar to the proof of Lemma 3.3. Suppose that $\partial_- C$ has a c-disc P . As in the previous lemma, there is no sc-disc D for P such that ∂D bounds an unpunctured disc on P but $|D \cap T| = 1$. Consequently, compressing P using any sc-disc D creates a new disc P' intersecting T the same number of times as P and with $\partial P' = \partial P$. Since ∂D is essential on $\partial_- C \setminus T$, the disc P' is a c-disc for $\partial_- C$. Thus, we may assume that P is disjoint from a complete collection of discs for (C, T) . It follows easily that P is ∂ -parallel in (C, T) and so is not a c-disc for $\partial_- C$, contrary to our assumption.

Proof of (3): Suppose that D is an sc-disc for $\partial_+ C$. Let Δ be a complete collection of discs for $\partial_+ C$ chosen so as to intersect D minimally. If $D \cap \Delta = \emptyset$, then D is either parallel to a disc of Δ or can be added to Δ to create another complete collection of discs for $\partial_+ C$. Assume, therefore, that $D \cap \Delta \neq \emptyset$.

By Lemma 3.3, a curve $\zeta \subset D \cap \Delta$ bounds a once-punctured disc in D if and only if it bounds a once-punctured disc in Δ . One-after-the-other, use circles in $D \cap \Delta$ which are innermost in D to compress Δ and then isotope to minimize intersections with Δ . The result is a disc D' and spheres P_i for $i \in \{1, \dots, n\}$. The proof of Lemma 3.3, shows that each of these spheres is parallel to a component of $\partial_- P$. An easy argument shows that these spheres cannot be nested, so for each component of $\partial_- C$, there is at most one sphere P_i parallel to it in (C, T) .

Consider now an arc $\zeta \subset D' \cap \Delta$. In Δ , the arc ζ cuts one of the discs of Δ into two discs with one, possibly containing a point of intersection with T . Thus, if $D' \cap \Delta \neq \emptyset$, then there is an arc $\zeta \subset D' \cap \Delta$ which is outermost in Δ and bounds an outermost disc in Δ disjoint from T . Repeatedly ∂ -compress D' using outermost discs of $\Delta \cap D'$ to obtain discs D'_1, \dots, D'_m , each disjoint from Δ and, altogether, intersecting T the same number of times as D .

Let (C', T') be the union of trivial v.p.-compressionbodies which results from reducing (C, T) using Δ . The discs and spheres $D'_1, \dots, D'_m, P_1, \dots, P_n \subset (C', T')$ are all ∂ -parallel. Boundary-reducing (C', T') using D'_i cuts off a (B^3, \emptyset) except for at most one $i' \in \{1, \dots, m\}$ which cuts off a (B^3, arc) . In $\partial_+ C'$ are the remnants of the discs of Δ . Examining $\partial_+ C$, we see that we can then isotope the remnants of Δ to be disjoint from the unpunctured discs in $\partial_+ C$ parallel to D'_i for $i \neq i'$. That is, we may isotope the discs Δ so that $D' \cap \Delta = \emptyset$.

Without loss of generality, assume that we can recover D from D' by tubing D' first to P_1 , then to P_2 , etc. If D' and P_1 lie in different components C'_0 and C'_1 , respectively, of C' , we can start to recover C from C' by attaching a tube $D^2 \times I$, possibly with $\{(0, 0)\} \times I \subset T$, from $\partial_+ C'_0$ to $\partial_+ C'_1$. A nested tube can then be attached to tube together D' and P'_1 . Isotoping $\partial D'$ across the tube moves D' into C'_1 . Equivalently, by rechoosing Δ , we may assume that D' and P_1 lie in the same

component of C' . We continue on in this vein, to show that Δ may be chosen to be disjoint from D , concluding the lemma. \square

Remark 3.5. It is not necessarily the case that if (C, T) is an irreducible v.p.-compressionbody then there is no sc-disc for $\partial_- C$. To see this, let (C, T) be the result of removing the interior of a regular neighborhood of a point on a vertical arc in an irreducible v.p.-compressionbody (\tilde{C}, \tilde{T}) . Then there is an sc-disc for $\partial_- C$ which is boundary-parallel in $\tilde{C} \setminus \tilde{T}$ and which cuts off from (C, T) a compressionbody which is $S^2 \times I$ intersecting T in two vertical arcs. See the top diagram in Figure 5. Similarly, if $\partial_- C$ contains 2-spheres disjoint from T , there will be a semi-compressing disc for any component of $\partial_- C$ which is not a 2-sphere disjoint from T .

We also cannot drop the hypothesis that no component of $\partial_- C$ is a sphere intersecting T exactly once. The bottom diagram in Figure 5 shows an sc-disc with the property that boundary-reducing the v.p.-compressionbody along that disc does not result in the union of v.p.-compressionbodies.

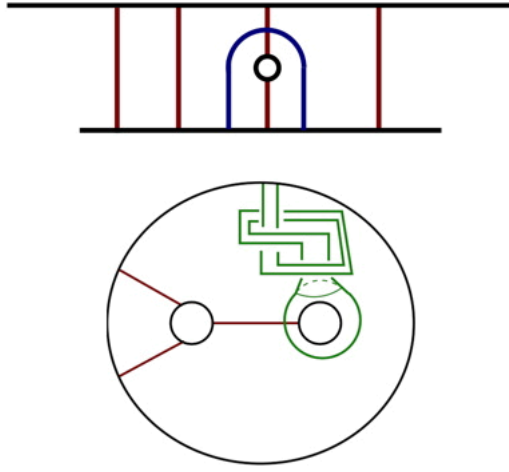


FIGURE 5. Above is an example of a v.p. compressionbody (C, T) where $\partial_- C$ has a semi-cut disc (shown in blue). Below is an example of an sc-disc with the property that boundary-reducing the v.p.-compressionbody along that disc does not result in the union of v.p.-compressionbodies.

In what follows, we will often use Lemma 3.4 without comment.

3.2. Multiple v.p. bridge surfaces. The definition of a multiple v.p.-bridge surface for the pair (M, T) which we are about to present is a version of Scharlemann and Thompson’s “generalized Heegaard splittings” [13] in the style of [5], but using v.p.-compressionbodies. We will also make use of orientations and height functions in a similar way to what shows up in Gabai’s definition of thin position [3] and the definition of Johnson’s “complex of surfaces” [6].

Definition 3.6. A connected closed surface $H \subset (M, T)$ is a **v.p.-bridge surface** for the pair (M, T) if $(M, T) \setminus H$ is the union of two distinct v.p.-compressionbodies (H_\uparrow, T_\uparrow) and $(H_\downarrow, T_\downarrow)$ with $H = \partial_+ H_\uparrow = \partial_+ H_\downarrow$. If $T = \emptyset$, then we also call H a **Heegaard surface** for M .

A **multiple v.p.-bridge surface** for (M, T) is a closed (possibly disconnected) surface $\mathcal{H} \subset (M, T)$ such that:

- \mathcal{H} is the disjoint union of \mathcal{H}^- and \mathcal{H}^+ , each of which is the union of components of \mathcal{H} ;
- $(M, T) \setminus \mathcal{H}$ is the union of embedded v.p.-compressionbodies (C_i, T_i) with $\mathcal{H}^- \cup \partial M = \bigcup \partial_- C_i$ and $\mathcal{H}^+ = \bigcup \partial_+ C_i$;
- Each component of \mathcal{H} is adjacent to two distinct compressionbodies.

If $T = \emptyset$, then \mathcal{H} is also called a **multiple Heegaard surface** for M . The components of \mathcal{H}^- are called **thin surfaces** and the components of \mathcal{H}^+ are called **thick surfaces**. We denote the set of multiple v.p.-bridge surfaces for (M, T) by $vp\mathbb{H}(M, T)$.

Note that each component of \mathcal{H}^+ is a v.p.-bridge surface for the component of $(M, T) \setminus \mathcal{H}^-$ containing it. In particular, if $H \in vp\mathbb{H}(M, T)$ is connected, then it is a v.p.-bridge surface and $H^- = \emptyset$. We now introduce orientations and height functions.

Definition 3.7. Suppose that \mathcal{H} is a multiple v.p.-bridge surface for (M, T) . Suppose that each component of \mathcal{H} is given a transverse orientation so that the following hold:

- If (C, T_C) is a component of $(M, T) \setminus \mathcal{H}$ then the transverse orientations of the components of $\partial_- C$ either all point into or all point out of C .
- If (C, T_C) is a component of $(M, T) \setminus \mathcal{H}$, then if the transverse orientation of $\partial_+ C$ points into (respectively, out of) C , then the transverse orientations of the components of $\partial_- C$ point out of (respectively, into) C .

If S_1 and S_2 are components of \mathcal{H} , then a **flow line** from S_1 to S_2 is an oriented path in M transverse to \mathcal{H} which starts at S_1 , ends at S_2 , and which, whenever it intersects a component S of \mathcal{H} does so in a direction consistent with the normal orientation on S . The multiple v.p.-bridge surface \mathcal{H} is an **oriented** multiple v.p.-bridge surface if each component of \mathcal{H} has a transverse orientation as above and if there is a function $f: \mathcal{H} \rightarrow \mathbb{N}$ such that the following hold:

- If there is a component S of \mathcal{H} such that $x, y \in S$, then $f(x) = f(y)$.
- If S_1 and S_2 are components of \mathcal{H} such that there is a flow line from S_1 to S_2 , then $f(S_1) < f(S_2)$.

The function f is called the **height function** for \mathcal{H} . The set of oriented multiple v.p.-bridge surfaces for (M, T) is denoted $\overrightarrow{vp\mathbb{H}}(M, T)$. We will always consider $\overrightarrow{vp\mathbb{H}}(M, T)$ to be a subset of $vp\mathbb{H}(M, T)$ via the map which “forgets” the orientation and height function. We say that $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ is **linear** if each component of \mathcal{H}^+ separates M .

See Figure 6 for two depictions of oriented multiple v.p.-bridge surfaces. Not all multiple v.p.-bridge surfaces can be oriented. For example, circular thin position (defined in [8]) clearly cannot have a concept of “above”. Notice, however, that every connected v.p.-bridge surface, once it is given a transverse orientation and height function, is a linear, oriented multiple v.p.-bridge surface.

Finally, for this section, we observe that cutting open along a thin surface induces oriented multiple bridge surfaces of the components.

Lemma 3.8. Suppose that $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ and that $F \subset \mathcal{H}^-$ is a component. Let (M', T') be a component of $(M, T) \setminus F$ and let $\mathcal{K} = (\mathcal{H} \setminus F) \cap M'$. Then $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M', T')$ and if \mathcal{H} was linear, so is \mathcal{K} .

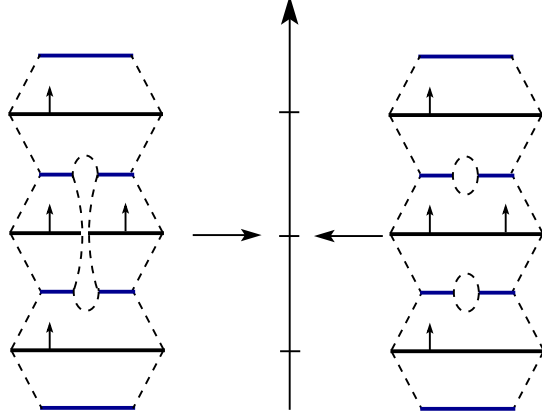


FIGURE 6. Two examples of oriented bridge surfaces. The one on the right is linear.

The proof of Lemma 3.8 follows immediately from the definitions, as the orientation and height function on \mathcal{H} restrict to an orientation and height function on \mathcal{K} .

4. SIMPLIFYING BRIDGE SURFACES

This section presents a host of ways of replacing certain types of multiple v.p.-bridge surfaces by new ones that are closely related but are “simpler” (we will make this concept precise in section 6). These simplifications are similar to the notion of “destabilization” and “weak reduction” for Heegaard splittings. Versions of many of these have appeared in other papers (e.g., [5, 9, 16–18].) The operations are: (generalized) destabilization, unperturbing, undoing a removable arc, untelescoping, and consolidation.

4.1. Destabilizing. Given a Heegaard splitting one can always obtain a Heegaard splitting of higher genus by adding a cancelling pair of a one-handle and a two-handle, or (if the manifold has boundary) by tubing the Heegaard surface to the frontier of a collar neighborhood of a component of the boundary of the manifold. In the case where the manifold contains a graph, the core of the 1-handle, the co-core of the 2-handle, or the core of the tube might be part of the graph. (Though in this paper, we do not need to consider the case when *both* the 1-handle and the 2-handle contain portions of the graph.) In the realm of Heegaard splittings the higher genus Heegaard splitting is said to be either a stabilization or a boundary-stabilization of the lower genus one. Observe that drilling out edges of T disjoint from $\mathcal{H} \in vp\mathbb{H}(M, T)$ preserves the fact that \mathcal{H} is a multiple v.p.-bridge surface. This suggests we also need to consider boundary-stabilization along portions of the graph T . Without further ado, here are our versions of destabilization:

Definition 4.1. Suppose that $\mathcal{H} \in vp\mathbb{H}(M, T)$ and let H be a component of \mathcal{H}^+ . There are six situations in which we can replace H by a new thick surface H' that is obtained from H by an compressing along an sc-disc. If H satisfies any of these conditions we say that H and \mathcal{H} contain a **generalized stabilization**. See Figure 7 for examples.

- There is a pair of compressing discs for H which intersect transversally in a single point and are contained on opposite sides of H and in the complement of all other surfaces of \mathcal{H} . In this case we say that H and \mathcal{H} are **stabilized**. The pair of compressing discs is called a **stabilizing pair**. The surface H' is obtained from H by compressing along either of the discs.

- There is a pair of a compressing disc and a cut disc for H which intersect transversally in a single point and are contained on opposite sides of H and in the complement of all other surfaces of \mathcal{H} . In this case we say that H and \mathcal{H} are **meridionally stabilized**. The pair of compressing disc and cut disc is called a **meridional stabilizing pair**. The surface H' is obtained by compressing H along the cut disc.
- There is a separating compressing disc D for H contained in the complement of all other surfaces of \mathcal{H} such that the following hold. Let W be the component of $M \setminus \mathcal{H}^-$ containing H . Compressing H along D produces two connected surfaces, H' and H'' , where H' is a v.p.-bridge surface for W and H'' bounds a trivial product compressionbody disjoint from H' with a component S of ∂M . In this case we say that H and \mathcal{H} are **boundary-stabilized** along S .
- There is a separating cut disc D for H contained in the complement of all other surfaces of \mathcal{H} such that the following hold. Let W be the component of $M \setminus \mathcal{H}^-$ containing H . Compressing H along D produces two connected surfaces, H' and H'' , where H' is a v.p.-bridge surface for W and H'' bounds a trivial product compressionbody disjoint from H'' with a component S of ∂M . In this case we say that H and \mathcal{H} are **meridionally boundary-stabilized** along S .
- Let G be a non-empty collection of vertices and edges of T disjoint from \mathcal{H} . Let $\widetilde{M} = M \setminus G$. If H and \mathcal{H} as a multiple v.p. bridge surface of \widetilde{M} are (meridionally) boundary stabilized along a component of $\partial \widetilde{M}$ which is not a component of ∂M , then H and \mathcal{H} are **(meridionally) ghost boundary-stabilized** along G .

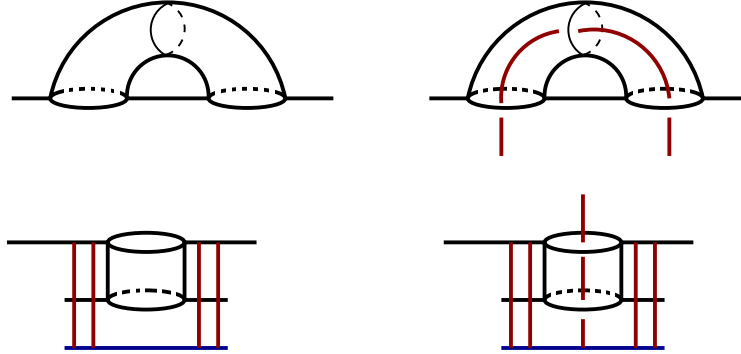


FIGURE 7. Depictions of stabilization, meridional stabilization, boundary stabilization, and meridional boundary stabilization.

Remark 4.2. Suppose that $H \subset \mathcal{H}^+$ has a generalized stabilization and let H' be the surface obtained from H by sc-compressing as in the definition above. It is easy to check (as in the classical settings) that $\mathcal{K} = (\mathcal{H} \setminus H) \cup H'$ is a multiple v.p.-bridge surface for (M, T) . If $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$, the transverse orientation on \mathcal{H} induces a transverse orientation on \mathcal{K} . If f is the height function on \mathcal{H} , then defining $f(H') = f(H)$ makes f a height function on \mathcal{K} . In particular, if $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$, there is a natural way of thinking of \mathcal{K} as an element of $\overrightarrow{vp\mathbb{H}}(M, T)$. Observe that if \mathcal{H} was linear, then \mathcal{K} is as well. We say that the (oriented) multiple v.p.-bridge surface \mathcal{K} is obtained by **destabilizing** \mathcal{H} (and that the thick surface H' is obtained by **destabilizing** the thick surface H .) Observe also that if \mathcal{H} was linear, then \mathcal{K} is as well.

4.2. Perturbed and Removable Bridge Surfaces. We can sometimes push a bridge surface across a bridge disc and obtain another bridge surface. This operation is called unperturbing.

Definition 4.3. Let $\mathcal{H} \in vp\mathbb{H}(M, T)$ and let $H \subset \mathcal{H}^+$ be a component. Suppose that there are bridge discs D_1 and D_2 for H in $M \setminus \mathcal{H}^-$, on opposite sides, disjoint from the vertices of T , and which have the property that the arcs $\alpha_1 = \partial D_1 \cap H$ and $\alpha_2 = \partial D_2 \cap H$ share exactly one endpoint and have disjoint interiors. Then H and \mathcal{H} have a **perturbation**. The discs D_1 and D_2 are called a **perturbing pair** of discs for H and \mathcal{H} .

Remark 4.4. The type of perturbation we have defined here might better be called an “arc-arc”-perturbation. There are also perturbations where the bridge discs are allowed to contain vertices of T , but we will not need them in this paper.

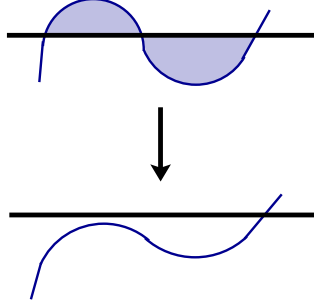


FIGURE 8. Unperturbing H .

Lemma 4.5. Let \mathcal{H} be an (oriented) multiple v.p.-bridge surface for (M, T) . Suppose that $H \subset \mathcal{H}^+$ is a perturbed component with perturbing discs D_1 and D_2 . Let D be the frontier of the neighborhood of D_1 . Then compressing H along D and discarding the resulting twice punctured sphere component results in a new surface H' and $\mathcal{K} = (\mathcal{H} - H) \cup H'$ is an (oriented) multiple v.p.-bridge surface for (M, T) . Furthermore, if \mathcal{H} is linear, then so is \mathcal{K} .

Proof. We can alternatively think of H' as obtained from H by an isotopy along D_1 . As this isotopy is supported in a ball containing a single arc of the tangle and no vertices we only need to show that the new arc contained on one side of H' is a bridge arc. A bridge disc for this arc can be constructed exactly as in Lemma 3.1 of [16]. If \mathcal{H} is oriented we make \mathcal{K} oriented by using the transverse orientations induced from \mathcal{H} and we let the height function on \mathcal{K} be the obvious choice induced from the height function on \mathcal{H} . \square

We say that the (oriented) multiple v.p. bridge surface \mathcal{K} constructed in the proof is obtained by **unperturbing** \mathcal{H} . See Figure 8 for a schematic depiction of the unperturbing operation.

4.3. Removable Pairs. Suppose that \mathcal{H} is an (oriented) multiple v.p.-bridge surface for (M, T) such that no component of $\mathcal{H}^- \cup \partial M$ is a sphere intersecting T exactly once. Let H be a component of \mathcal{H}^+ , with \mathcal{D}_\uparrow and \mathcal{D}_\downarrow complete sets of discs for (H_\uparrow, T_\uparrow) and $(H_\downarrow, T_\downarrow)$ respectively. Suppose that there exists a bridge disc D for H in H_\uparrow (or H_\downarrow) with the following properties:

- it is disjoint from the vertices of T ;
- it is disjoint from \mathcal{D}_\uparrow (resp. \mathcal{D}_\downarrow);
- the arc $\partial D \cap H$ intersects a single component D^* of \mathcal{D}_\downarrow (resp. \mathcal{D}_\uparrow). D^* is a disc and $|D \cap D^*| = 1$,

then \mathcal{H} and H are **removable**. The discs D and D^* are called a **removing pair**. See the left side of Figure 9.

Example 4.6. Suppose that $H \in \text{vp}\mathbb{H}(M, T)$ is connected and that M' is obtained from M by attaching a 2-handle to ∂M or Dehn-filling a torus component of ∂M . Let α be either a co-core of the 2-handle or a core of the filling torus. Using an unknotted path in $M - H$, isotope α so that it intersects H exactly twice. Then $H \in \text{vp}\mathbb{H}(M, T \cup \alpha)$ is removable. The component α is called the **removable component of $T \cup \alpha$** .

Lemma 4.7. *Suppose that $\mathcal{H} \in \text{vp}\mathbb{H}(M, T)$ is removable. Then there is an isotopy of \mathcal{H} in M to $\mathcal{K} \in \text{vp}\mathbb{H}(M, T)$ supported in the neighborhood of the removing pair. Furthermore, if \mathcal{H} is oriented, so is \mathcal{K} and if \mathcal{H} is linear, so is \mathcal{K} .*

Proof. Let H be the thick surface which is removable. We will construct an isotopy from H to a surface H' supported in a regular neighborhood of the removing pair and let $\mathcal{K} = (\mathcal{H} - H) \cup H'$. We will show that \mathcal{K} is a v.p. multiple bridge surface. Assuming it is, if $\mathcal{H} \in \overrightarrow{\text{vp}\mathbb{H}}(M, T)$ with height function $f_{\mathcal{H}}$ we give H' the normal orientation induced by H and define $f_{\mathcal{K}}(H') = f_{\mathcal{H}}(H)$. It is then easy to show that $\mathcal{K} \in \overrightarrow{\text{vp}\mathbb{H}}(M, T)$ and if \mathcal{H} is linear then so is \mathcal{K} .

Without the loss of much generality, we may assume that $\mathcal{H} = H$ is connected.

Let $D \subset H_{\uparrow}$ and $D^* \subset H_{\downarrow}$ be the removing pair and let \mathcal{D}_{\uparrow} and \mathcal{D}_{\downarrow} be the corresponding complete set of discs from the definition of “removable”. Isotope T across D so that $T \cap D$ lies in H_{\downarrow} . Let T' be the resulting graph and let D_c^* be the cut disc that D^* gets converted into. Equivalently, we may isotope H across D and let H' be the resulting surface. See Figure 9.

The graph $T'_{\uparrow} = T' \cap H_{\uparrow}$ is obtained from $T_{\uparrow} = T \cap H_{\uparrow}$ by removing a component of T_{\uparrow} . After creating T' from T , the collection \mathcal{D}_{\uparrow} remains a set of discs that decompose $(H_{\uparrow}, T'_{\uparrow})$ into trivial compressionbodies, although there may now be discs in \mathcal{D}_{\uparrow} which are parallel or which are boundary-parallel in $H_{\uparrow} \setminus T'_{\uparrow}$. Thus, $(H_{\uparrow}, T'_{\uparrow})$ is a v.p.-compressionbody.

To show that $(H_{\downarrow}, T'_{\downarrow})$ is a v.p.-compressionbody, note that cut-compressing (H_{\downarrow}, T') along D_c^* results in the same collection of compressionbodies as compressing (H_{\downarrow}, T) along D^* . Therefore \mathcal{D}_{\downarrow} with D^* replaced by the induced cut disc D_c^* is a complete collection of sc-discs for $(H_{\downarrow}, T'_{\downarrow})$ and so $(H_{\downarrow}, T'_{\downarrow})$ is a v.p.-compressionbody. We conclude that \mathcal{K} is an (oriented, linear) multiple v.p.-bridge surface. \square

The surface \mathcal{K} in the preceding lemma is said to be obtained by **undoing a removable arc** of \mathcal{H} .

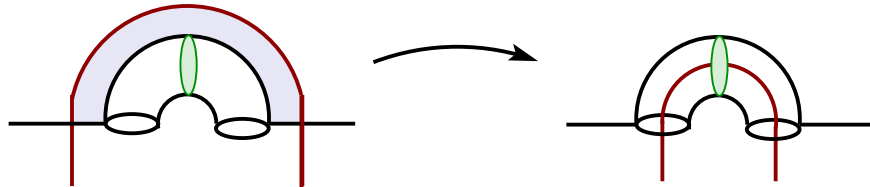


FIGURE 9. Undoing a removable arc

5. UNTELESCOPING AND CONSOLIDATION

If we let T be empty in everything discussed so far and if we ignore the height function we defined, then we are in Scharlemann-Thompson's set-up for thin position. We need a way to recognize when the multiple bridge surface can be "thinned" and a way to show that this thinning process eventually terminates. Scharlemann and Thompson thin by switching the order in which some pair (1-handle, 2-handle) are added and they use Casson-Gordon's criterion [2] to recognize that this is possible by finding disjoint compressing discs on opposite sides of a thick surface. In this section, we use compressions along sc-weak reducing pairs of discs in place of handle exchanges.

5.1. Untelescoping. Suppose that $\mathcal{H} \in \text{vp}\mathbb{H}(M, T)$. If \mathcal{H} has the property that there is a component $H \subset \mathcal{H}^+$, and disjoint sc-discs D_- and D_+ for H on opposite sides so that D_- and D_+ are disjoint from \mathcal{H}^- , we say that \mathcal{H} is **sc-weakly reducible**, that H is the **sc-weakly reducible component** and that $\{D_-, D_+\}$ is a **sc-weakly reducing pair**. If \mathcal{H} is not sc-weakly reducible, we say it is **sc-strongly irreducible**. If D_- and D_+ are c-discs, we also say that \mathcal{H} is **c-weakly reducible**, etc. Suppose that no component of $\mathcal{H}^- \cup \partial M$ is a sphere intersecting T exactly once. Then, given an sc-weakly reducible $\mathcal{H} \in \text{vp}\mathbb{H}(M, T)$, we can create a new $\mathcal{K} \in \text{vp}\mathbb{H}(M, T)$ by **untelescoping** \mathcal{H} as follows:

Definition 5.1. Let $\{D_-, D_+\}$ be an sc-weakly reducing pair for an sc-weakly reducible component H of \mathcal{H}^+ . Let N be the component of $M \setminus \mathcal{H}^-$ containing H . Let H_0 be the result of c-compressing H using both D_- and D_+ . Let H_\pm be the result of compressing H using only D_\pm and isotope each of H_\pm slightly into the compressionbody containing D_\pm , respectively. Let $\mathcal{K}^- = \mathcal{H}^- \cup H_0$ and $\mathcal{K}^+ = (\mathcal{H}^+ \setminus H) \cup (H_- \cup H_+)$. See Figure 10 for a schematic picture. The component of H_0 adjacent to copies of both D_- and D_+ is called the **doubly spotted** component. (The terminology is taken from [11].)

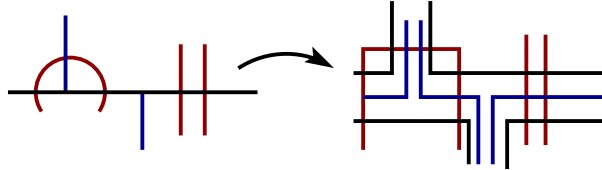


FIGURE 10. Untelescoping H . The red curves are portions of T . The blue lines on the left are sc-discs for H . Note that if a semi-cut or cut disc is used then a ghost arc is created.

Lemma 5.2. *If $\mathcal{H} \in \text{vp}\mathbb{H}(M, T)$ and if \mathcal{K} is obtained by untelescoping \mathcal{H} , then $\mathcal{K} \in \text{vp}\mathbb{H}(M, T)$.*

Proof. Let $H \subset \mathcal{H}^+$ be the component which is untelescoped using discs $\{D_-, D_+\}$. Let W_+ and W_- be the two compressionbody components of $M \setminus \mathcal{H}$ that have copies of H as their positive boundaries. Let \mathcal{D}_\pm be a complete collections of discs for the compressionbodies W_\pm containing D_\pm . The discs $\mathcal{D}_\pm \setminus D_\pm$ after an isotopy are a complete collection of discs for the components of $(W_- \cup W_+) \setminus \mathcal{K}$ not adjacent to $\mathcal{K}^- \setminus \mathcal{H}^-$. The components adjacent to \mathcal{K}^- are obtained from the trivial product compressionbodies by attaching a single 1-handle (dual to the 2-handles with cores D_\pm) and so $\mathcal{K} \in \text{vp}\mathbb{H}(M, T)$. \square

To extend this operation to oriented multiple v.p. bridge surfaces we need to specify how height functions behave under untelescoping.

Definition 5.3. Use the notation from Definition 5.1 and assume that $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$. Let $f_{\mathcal{H}}$ be the height function for \mathcal{H} . Suppose that $H_- \subset H_{\downarrow}$ and $H_+ \subset H_{\uparrow}$. Give the components of \mathcal{K} the induced transverse orientations from \mathcal{H} and define the height function $f_{\mathcal{K}}$ for \mathcal{K} as follows. Let $S \subset \mathcal{K}$ be a component.

- If S is a component of \mathcal{H} such that $f_{\mathcal{H}}(S) < f_{\mathcal{H}}(H)$, then $f_{\mathcal{K}}(S) = f_{\mathcal{H}}(S)$.
- If S is a component of H_- , then $f_{\mathcal{K}}(S) = f_{\mathcal{H}}(H)$.
- If S is a component of the thin surface between H_- and H_+ then $f_{\mathcal{K}}(S) = f_{\mathcal{H}}(H) + 1$
- If S is a component of H_+ then $f_{\mathcal{K}}(S) = f_{\mathcal{H}}(H) + 2$
- If S is a component of \mathcal{H} such that $S \neq H$ and $f_{\mathcal{H}}(S) \geq f_{\mathcal{H}}(H)$, then $f_{\mathcal{K}}(S) = f_{\mathcal{H}}(S) + 2$.

With Definition 5.3, it is easy to check that $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$. If \mathcal{H} was linear, it may not be the case that \mathcal{K} is linear. The simplifications described in the next section will, however, return linearity.

5.2. Consolidation. Untelescoping usually creates product compressionbodies which need to be removed as in Scharlemann-Thompson thin position. In our situation though this process is complicated by the presence of the graph T . We call the operation “consolidation.”

Definition 5.4. Suppose that \mathcal{H} is an (oriented) multiple v.p.-bridge surface for (M, T) and that (P, T_P) is a product compressionbody component of $(M, T) \setminus \mathcal{H}$ which is adjacent to a component of \mathcal{H}^- (and, therefore, not adjacent to a component of ∂M .) Let $\mathcal{K} = \mathcal{H} \setminus (\partial_- P \cup \partial_+ P)$. If \mathcal{H} is oriented, give each component of \mathcal{K} the induced orientation from \mathcal{H} . The height function on \mathcal{K} is defined to be the restriction of the height function on \mathcal{H} . We say that \mathcal{K} is obtained from \mathcal{H} by **consolidation** or by **consolidating** (P, T_P) . (These terms were introduced in [18].)

The next two lemmas verify that consolidation is a valid operation in $vp\mathbb{H}(M, T)$. See Figure 13 for a schematic depiction of the v.p.-compressionbodies in the following lemma.

Lemma 5.5. *Suppose that (P, T_P) is a trivial product compression body and that (A, T_A) and (B, T_A) are v.p.-compressionbodies with interiors disjoint from each other and from the interior of P . Assume also that $\partial_- P \subset \partial_- A$ and $\partial_+ B = \partial_+ P$. Let $(C, T) = (A, T_A) \cup (P, T_P) \cup (B, T_B)$ and assume that T is properly-embedded in C . Then (C, T) is a v.p.-compressionbody.*

Proof. We can dually define a v.p. compressionbody to be a 3-manifold containing a properly embedded 1-manifold obtained by taking a collection of trivial v.p. compressionbodies and adding to their positive boundary some 1-handles and 1-handles containing a single piece of tangle as their core. With this dual definition, the lemma is obvious. Alternatively, it is possible to prove it directly from the definition of v.p.-compressionbody. \square

Lemma 5.6. *Suppose that \mathcal{H} is an (oriented) multiple v.p.-bridge surface for (M, T) and that \mathcal{K} is obtained by consolidating a product region (P, T_P) of \mathcal{H} . Then \mathcal{K} is an (oriented) multiple v.p.-bridge surface for (M, T) . Furthermore, if \mathcal{H} is linear, so is \mathcal{K} .*

Proof. This follows immediately from Lemma 5.5 and the definitions. \square

5.3. Elementary Thinning Sequences. As mentioned before, untelescoping often produces product regions. These product regions, in general, are of two types – they can be between a thin and thick surface neither of which existed before the untelescoping or they can be between a newly created thick surface and a thin surface (or a boundary component) that existed before the untelescoping operation. In fact, consolidating product regions of the first type can create additional

product regions of the second type. The next definition specifies the order in which we will consolidate, before untelescoping further.

Definition 5.7. Suppose that \mathcal{H} is an sc-weakly reducible oriented multiple v.p.-bridge surface for (M, T) . Let \mathcal{H}_1 be obtained by untelescoping \mathcal{H} using an sc-weak reducing pair. Let \mathcal{H}_2 be obtained by consolidating all trivial product compressionbodies of $\mathcal{H}_1 \setminus \mathcal{H}$. There may now be trivial product compressionbodies in $M \setminus \mathcal{H}_2$. Let \mathcal{H}_3 be obtained by consolidating all those products. We say that \mathcal{H}_3 is obtained from \mathcal{H} by an **elementary thinning sequence**.

See Figure 11 for a depiction of the creation of \mathcal{H}_2 from \mathcal{H} .

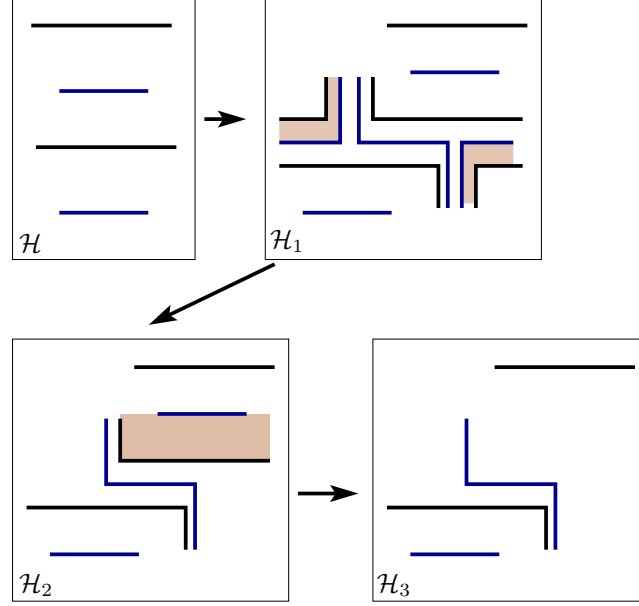


FIGURE 11. The surface \mathcal{H}_2 is created by untelescoping and consolidation. One or both of the compressionbodies $M \setminus \mathcal{H}_2$ shown in the figure may be product regions adjacent to \mathcal{H}^- . We consolidate those product regions to obtain \mathcal{H}_3 .

To understand the effect of an elementary thinning sequence, we examine the untelescoping operation a little more carefully.

Lemma 5.8. *Suppose that H is a connected (oriented) v.p.-bridge surface and that D_\uparrow and D_\downarrow are an sc-weak reducing pair. Let $H_- \subset H_\downarrow$ and $H_+ \subset H_\uparrow$ be the new thick surfaces created by untelescoping H . Let F be the thin surfaces. Then the following are equivalent for a component Φ of F :*

- (1) Φ is not doubly spotted and is adjacent to a remnant of D_\uparrow (or D_\downarrow , respectively).
- (2) The disc D_\uparrow (or D_\downarrow , respectively) is separating and Φ bounds a product region in H_\uparrow (or H_\downarrow , respectively) with a component of H_+ (or H_- , respectively.)

Proof. Suppose Φ is only adjacent to D_\uparrow . In this case D_\uparrow must be separating as otherwise Φ would have two spots from D_\uparrow , and as H is connected, it would also have to have a spot coming from D_\downarrow . Compressing H along D_\uparrow then results in two components. Let H' be the component that doesn't

contain ∂D_\downarrow . Then H' is not affected by compressing along D_\downarrow to obtain F . Thus H' is parallel to Φ .

Conversely if Φ is parallel to some component H' of H_+ say, then Φ must be disjoint from the compressing disc D_\downarrow and is therefore not double spotted. \square

Using the notation from Definition 5.7, we have:

Lemma 5.9. *Suppose that $\mathcal{H} \in \text{vp}\mathbb{H}(M, T)$ and that $(M, T) \setminus \mathcal{H}$ has no trivial product compressionbodies adjacent to \mathcal{H}^- . Let $\mathcal{H}_1, \mathcal{H}_2$, and \mathcal{H}_3 be the surfaces in an elementary thinning sequence beginning with the untelescoping of a component $H \subset \mathcal{H}^+$. Then the doubly spotted component of \mathcal{H}_1 persists into \mathcal{H}_3 and no component of $(M, T) \setminus \mathcal{H}_3$ is a trivial product compressionbody adjacent to \mathcal{H}_3^- .*

Proof. Let H_- and H_+ be the thick surfaces resulting from untelescoping the thick surface $H \subset \mathcal{H}^+$ and let F be the thin surface, with F_0 the doubly spotted component. Since F is obtained by compressing using an sc-disc, F is not parallel to either of H_- or H_+ . In creating \mathcal{H}_2 we remove all components of F which are not doubly spotted (Lemma 5.8). The doubly spotted surface is not parallel to the remaining components of H_- or H_+ since we can obtain it by an sc-compression of each of them. Thus, the doubly spotted component persists into \mathcal{H}_2 . Let H'_- and H'_+ be the components of H_- and H_+ remaining in \mathcal{H}_2 . If either of H_- or H_+ bounds a trivial product compressionbody with \mathcal{H}^- , we create \mathcal{H}_3 by consolidating those trivial product compressionbodies.

Suppose that a component $(W, T_W) \subset (M, T) \setminus \mathcal{H}_3$ contains $F \subset \partial_- W$. Since H_- and H_+ each had an sc-compression producing the doubly-spotted components of F , (W, T_W) must contain an sc-disc for $\partial_+ W$. Consequently, (W, T_W) is not a trivial product compressionbody. The result then follows from the assumption that no component of $(M, T) \setminus \mathcal{H}$ was a trivial product compressionbody adjacent to \mathcal{H}^- . \square

Corollary 5.10. *Suppose that \mathcal{H}, \mathcal{K} are multiple v.p.-bridge surfaces for (M, T) such that $M \setminus \mathcal{H}$ has no trivial product compressionbodies adjacent to \mathcal{H}^- . Assume that \mathcal{K} is obtained from \mathcal{H} using an elementary thinning sequence. Then the following are true:*

- (1) $\mathcal{K}^- \neq \emptyset$
- (2) \mathcal{K} has no trivial product compressionbodies disjoint from ∂M ,
- (3) If \mathcal{H} is linear, so is \mathcal{K} .

Proof. By Lemma 5.9, the doubly spotted component of $\mathcal{K}^- \setminus \mathcal{H}^-$ does not get consolidated during the elementary thinning sequence and \mathcal{K} has no trivial product compressionbodies adjacent to \mathcal{K}^- . It remains to show that if \mathcal{H} is linear, then so is \mathcal{K} .

Let H be the component of \mathcal{H}^+ which is untelescoped in the elementary thinning sequence. By the definition of “linear”, H is separating. Let \mathcal{H}_1 be the result of untelescoping using the sc-weak reducing pair $\{D_\downarrow, D_\uparrow\}$ with H_- and H_+ the new thick surfaces and F the new thin surfaces. Let \mathcal{H}_2 be the result of consolidating any product regions between H_- and F and between H_+ and F . By Lemma 5.8, a single component H'_\pm of H_\pm persists into \mathcal{H}_2 . Let F' be the union of the components of F which persist into \mathcal{H}_2 . We will show that \mathcal{H}_2 is linear by showing that H'_- and H'_+ are separating. It then follows from Lemma 5.6, that \mathcal{K} is linear.

The argument for H'_+ is the same as for H'_- , so suppose, to encounter a contradiction, that H'_- is non-separating. Let γ be a simple closed curve in M which intersects H'_- in a unique point. Suppose, first that $H'_- = H_-$. In this case, the disc D_- is non-separating and we may recover H

from H'_- by attaching a tube. By general position, we may assume the tube is disjoint from γ , in which case H intersects γ in a unique point. This contradicts the assumption that H is separating.

Thus, H_- has two components. That is, D_- is separating. One component is H'_- . Let the other be H''_- . Since H was separating, the loop γ must also intersect H''_- . Without loss of generality, we may assume it does so in a unique point. As before, we may recover H from H_- by attaching a tube from H'_- to H''_- which is disjoint from γ . Let γ' be the component of $\gamma \setminus H$ which lies on the same side of H as D_- . Observe that $\gamma' \cap D_- = \emptyset$. Since every component of \mathcal{H}^+ separates M , we may assume that γ' is disjoint from $\mathcal{H}^+ \setminus H$. Without loss of generality, we may also assume that it is disjoint from \mathcal{H}^- , as v.p.-compressionbodies are connected. Then the arc γ' lies in the v.p.-compressionbody below H , joins H'_- to H''_- and is disjoint from D_- . But this contradicts the assumption that D_- is separating. Thus, H'_- separates M . \square

6. COMPLEXITY

The theory of 3-manifolds is rife with various complexity functions on surfaces which guarantee certain processes terminate. In [13] Scharlemann and Thompson used a version of Euler characteristic as their measure of complexity to ensure that untelescoping (and consolidation) of Heegaard surfaces will eventually terminate. Since that foundational paper similar complexities have been used by many authors, eg. [5, 6]. The requirement for complexity is that it decreases under all possible types of compressions and any other moves that “should” simplify the decomposition. In our context, we need a complexity that decreases under destabilizing a generalized stabilization, unperturbing, undoing a removable arc, and applying an elementary thinning sequence.

If we were to restrict ourselves to c-discs when untelescoping, then we can use a complexity similar to Scharlemann-Thompson’s, as in the next definition. (This complexity is, in some sense, similar to a “rational bridge number” suggested by Lickorish [7, Problem 2].)

Definition 6.1. Suppose that $S \subset (M, T)$ is a surface. For fixed natural numbers $q_1 \geq 2$ and $q_2 \geq 1$, define the (q_1, q_2) -**complexity** of S to be

$$c(S) = -q_1 q_2 \chi(S) + q_2 |S \cap T| + 2q_1 q_2$$

where $\chi(S)$ is the Euler characteristic of the unpunctured surface. For a surface $\mathcal{H} \in vp\mathbb{H}(M, T)$ we define the **unoriented complexity** $\mathbf{c}(\mathcal{H})$ to be the multiset with elements $c(H)$ for all components $H \subset \mathcal{H}^+$. We compare complexities by arranging the elements in non-decreasing order and then using lexicographic ordering.

It is possible to show that this complexity decreases under destabilization, unperturbation, undoing a removable arc and an elementary thinning sequence, *as long as c-discs* and not semi-compressing or semi-cut discs are used. To be able to use semi-compressing and semi-cut discs, however, we need a more sophisticated construction. Before we give it, we exhibit an example as further motivation for using semi-compressing and semi-cut discs.

6.1. An example. Traditionally thin position in the style of Scharlemann-Thompson [13] is done only for irreducible 3-manifolds. However, the following example shows that, at an informal level, it should be possible to define a thin position for reducible 3-manifolds.

Let P be the result of removing a regular neighborhood of two points from a 3-ball. Choose one component of ∂P as $\partial_+ P$ and the other two as $\partial_- P$. Let M be the result of gluing two copies of P along $H = \partial_+ P$. Then there is a certain sense in which the splitting of M can be untelescoped to a simpler splitting, but the complexity of the thick surface increases. Figure 12 shows the original

Heegaard surface and another, ostensibly thinner, multiple Heegaard surface. The surface on the right can be obtained from the one on the left by thinning using semi-compressing discs.

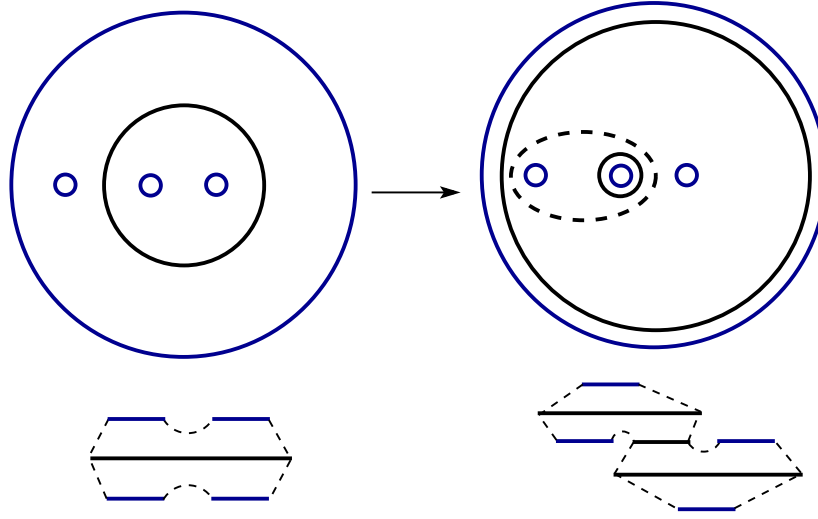


FIGURE 12. On the left in black is the original Heegaard surface and on right is a multiple Heegaard surface which “should”, by all rights, be thinner, except that it has larger complexity. The thick surfaces are in solid black and the thin surface is in dashed black. Below each figure is a schematic representation with the boundary components in blue, the thick surfaces in long black lines, and the thin surface in a short black line.

Although this example concerned a reducible manifold, we will run into similar problems when we have thin surfaces which are spheres twice-punctured by the graph T . If, in the example on the left, we add in a single ghost arc on each side of the Heegaard surface and four vertical arcs, one adjacent to each boundary component, we obtain an irreducible pair (M, T) with a connected v.p.-bridge surface that can be thinned to the surface on the right using semi-cut discs. Observe that neither of the v.p.-compressionbodies in the example on the left contains a compressing disc or a cut disc and that neither is a trivial v.p.-compressionbodies.

More generally, a Scharlemann-Thompson style complexity will not be helpful for thinning using semi-compressing or semi-cut discs. Indeed, if H is a thick surface such that $c(H)$ occurs in the multiset for a Scharlemann-Thompson complexity, then if H' is thinned using a semi-compressing or semi-cut disc we must replace $\{c(H)\}$ in the multiset with $\{c(H), c(H)\}$, increasing the complexity.

To rule out having to deal with such considerations, we introduce a new definition of complexity for oriented multiple v.p. bridge surfaces based on the idea that we can thin a handle decomposition by breaking the handle attachments into steps where in each step we attach as few 1-handles and as many 2-handles as possible. Rather than developing the appropriate handle-theory, we use the index of v.p.-compressionbodies.

6.2. Index of v.p.-compressionbodies. In order to develop a useful complexity for oriented multiple v.p.-bridge surfaces, we replace the complexity of surfaces with the index of compressionbodies. The index of a compressionbody without an embedded graph was first defined by Scharlemann and Schultens [12].

Definition 6.2. Let (C, T) be a v.p.-compressionbody such that T has no vertices. Define the **pre-index** of (C, T) to be

$$\mu^*(C, T) = 4(g(\partial_+ C) - g(\partial_- C)) + (|\partial_+ C \cap T| - |\partial_- C \cap T|) + 3|\partial_- C| - 3$$

The **index** of (C, T) is defined to be

$$\mu(C, T) = \max(0, \mu^*(C, T)).$$

If (C, T) is a v.p.-compressionbody such that T has vertices, define the pre-index and index of (C, T) to be the pre-index and index (respectively) of the v.p.-compressionbody obtained by drilling out the vertices of T . If (C, T) is the union of v.p.-compressionbodies with disjoint interiors, define the pre-index $\mu^*(C, T)$ and the index $\mu(C, T)$ to be the sum of the pre-indices and indices of the components of (C, T) , respectively.

If $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$, then the **total index** $\mu(\mathcal{H})$ of \mathcal{H} is the sum of the indices of all the components of $(M, T) \setminus \mathcal{H}$.

Remark 6.3. As we shall see in the proof of Lemma 6.4, the number $\mu(C, T)$ is related to the number of sc-discs in a complete set of discs for (C, T) . Other definitions of index can be used. For example, any number of the form

$$\mu^*(C, T) = a(g(\partial_+ C) - g(\partial_- C)) + b(|\partial_+ C \cap T| - |\partial_- C \cap T|) + c|\partial_- C| - c$$

with $a, b, c \in \mathbb{N}$ and $a > \max(c, 2b + c/2)$ and $4b > c > 2b$ will work in what follows.

We first establish an elementary bound on the difference between index and pre-index.

Lemma 6.4. *Suppose that (C, T) is a v.p.-compressionbody. Then $\mu^*(C, T) \geq -3$ with $\mu^*(C, T) < 0$ if and only if (C, T) is either (B^3, \emptyset) or (B^3, arc) . Furthermore, $\mu^*(C, T) = 0$ if and only if after drilling out the vertices of T , (C, T) is a trivial product compressionbody. Consequently, $\mu(C, T) = 0$ if and only if (C, T) is a trivial v.p.-compressionbody.*

Proof. It is easy to calculate that $\mu^*(B^3, \emptyset) = -3$ and $\mu^*(B^3, \text{arc}) = -1$ and that both the index and pre-index of a trivial product compressionbody are zero.

Without loss of generality, we may assume that T has no vertices (drilling them out, if necessary). We show that if $\mu(C, T) = 0$ then (C, T) is a trivial product compressionbody. Recall that T is the union of bridge arcs, vertical arcs, and ghost arcs. Also recall that, by the definition of compressionbody, $\partial_+ C$ is obtained by attaching tubes to $\partial_- C$ so that $g(\partial_+ C) \geq g(\partial_- C)$. Throughout, we let:

$$\begin{aligned} \mu^* &= \mu^*(C, T), \\ g_{\pm} &= g(\partial_{\pm} C), \text{ and} \\ |\partial_{\pm} T| &= |\partial_{\pm} C \cap T|. \end{aligned}$$

Case 1: T has no ghost arcs.

In this case, $|\partial_+ T| \geq |\partial_- T|$, and so:

$$\begin{aligned} \mu^* &= 4(g_+ - g_-) + (|\partial_+ T| - |\partial_- T|) + 3|\partial_- C| - 3 \\ &\geq 3|\partial_- C| - 3. \end{aligned}$$

Suppose that $\mu^* \leq 0$. Then, $|\partial_- C| \leq 1$. If $|\partial_- C| = 0$, then we must have $g_+ = 0$. Since C is a compressionbody, this implies that C is a 3-ball and $|\partial_+ T| = 2$. Thus, (C, T) is either $\mu^*(B^3, \emptyset)$ or $\mu^*(B^3, \text{arc})$, as desired. If $|\partial_- C| = 1$, then $g_+ = g_-$, and $|\partial_+ T| = |\partial_- T|$. This implies that (C, T) is a trivial product compressionbody.

Case 2: T has at least $g \geq 1$ ghost arcs.

We will show that $\mu^*(C, T) \geq 1$. The hypothesis implies that $|\partial_- C| \geq 1$ and that (C, T) is not a trivial v.p.-compressionbody. Let b be the number of bridge arcs in T and let g be the number of ghost arcs in T . By the definition of v.p.-compressionbody, we have:

$$\begin{aligned} |\partial_+ T| - |\partial_- T| &= 2b - 2g, \text{ and} \\ g_+ - g_- &\geq g - |\partial_- C| + 1. \end{aligned}$$

Hence,

$$(1) \quad (g_+ - g_-) + |\partial_- C| \geq g + 1.$$

Using Inequality (1) at the first instance of \geq below, we have:

$$\begin{aligned} \mu^* &= 3(g_+ - g_-) + 2b - 2g + 3|\partial_- C| + (g_+ - g_-) - 3 \\ &= 3((g_+ - g_-) + |\partial_- C|) + 2b - 2g + (g_+ - g_-) - 3 \\ &\geq 3(g + 1) + 2b - 2g + (g_+ - g_-) - 3 \\ &= g + 2b + (g_+ - g_-) \\ &\geq 1. \end{aligned}$$

Thus, $\mu^*(C, T) \geq 1$. □

Lemma 6.5. *Suppose that (C, T) is a v.p.-compressionbody and that $D \subset (C, T)$ is a disc with $\partial D \subset \partial_+ C$ and with $|D \cap T| \leq 1$. Let (C', T') be the result of reducing (C, T) using D . Let $p = 2|D \cap T|$. Assuming (C', T') is a v.p.-compressionbody, then the following hold:*

(1) *If D is separating, then*

$$\mu^*(C', T') = \mu^*(C, T) - 3 + p$$

(2) *If D is non-separating, then*

$$\mu^*(C', T') = \mu^*(C, T) - 4 + p$$

Proof. Suppose, first, that D is separating and let (C_1, T_1) and (C_2, T_2) be the two components of (C', T') . Observe that:

$$\begin{aligned} g(\partial_\pm C) &= g(\partial_\pm C_1) + g(\partial_\pm C_2) \\ |\partial_+ C \cap T| &= |\partial_+ C_1 \cap T| + |\partial_+ C_2 \cap T| + p \\ |\partial_- C \cap T| &= |\partial_- C_1 \cap T| + |\partial_- C_2 \cap T| \\ |\partial_- C| &= |\partial_- C_1| + |\partial_- C_2|. \end{aligned}$$

Thus, by definition of μ^* :

$$\begin{aligned} \mu^*(C', T') &= \mu^*(C_1, T_1) + \mu^*(C_2, T_2) \\ &= 4(g(\partial_+ C) - g(\partial_- C)) \\ &\quad + (|\partial_+ C \cap T| - |\partial_- C \cap T|) + p \\ &\quad + 3|\partial_- C| - 6 \\ &= \mu^*(C, T) - 3 + p. \end{aligned}$$

Now suppose that D is non-separating. Observe that:

$$\begin{aligned} g(\partial_+ C') &= g(\partial_+ C) - 1 \\ |\partial_+ C' \cap T| &= |\partial_+ C \cap T| + p \\ \partial_- C' &= \partial_- C. \end{aligned}$$

Thus,

$$\begin{aligned}
\mu(C', T') &= 4(g(\partial_+ C) - g(\partial_- C)) - 4 \\
&+ |\partial_+ C \cap T| - |\partial_- C \cap T| + p \\
&+ 3|\partial_- C| - 3 \\
&= \mu^*(C, T) - 4 + p.
\end{aligned}$$

□

Corollary 6.6. *Suppose that (C, T) is a v.p.-compressionbody and that the v.p.-compressionbody (C', T') is the result of reducing (C, T) using an sc-disc D , with (C_1, T_1) a component of (C', T') . Then $\mu(C', T') < \mu(C, T)$ and $\mu(C_1, T_1) \leq \mu(C', T')$.*

Proof. Without loss of generality, we may assume that T does not have vertices. Since (C, T) has an sc-disc, it is not trivial. Hence, by Lemma 6.4, $\mu(C, T) > 0$. Consequently, $\mu(C, T) = \mu^*(C, T)$.

Suppose D is non-separating, then $(C_1, T_1) = (C', T')$. If (C', T') is a trivial compressionbody, then $\mu(C', T') = 0 < \mu(C, T)$ and we are done. So we may assume that $\mu(C', T') = \mu^*(C', T')$. By Lemma 6.5,

$$\mu(C_1, T_1) = \mu(C', T') \leq \mu(C, T) - 2 < \mu(C, T),$$

as desired.

Suppose that D is separating and that (C_1, T_1) and (C_2, T_2) are the two components. Neither can be (B^3, \emptyset) , as otherwise, D would be ∂ -parallel. If D is a (semi)-cut disc, then neither can be (B^3, arc) as otherwise D would be ∂ -parallel. Let $\delta_i = 1$ if (C_i, T_i) is (B^3, arc) and let $\delta_i = 0$ otherwise. Hence, $\mu(C_i, T_i) = \mu^*(C_i, T_i) + \delta_i$. If both (C_1, T_1) and (C_2, T_2) are trivial compressionbodies, then $\mu(C', T') = \mu(C_1, T_1) = 0 < \mu(C', T')$. Thus, we may assume that $\delta_1 + \delta_2 \in \{0, 1\}$.

Let $p = 2|D \cap T|$. Then, by Lemma 6.5,

$$\begin{aligned}
\mu(C', T') &= \mu(C_1, T_1) + \mu(C_2, T_2) \\
&= \mu^*(C_1, T_1) + \mu^*(C_2, T_2) + \delta_1 + \delta_2 \\
&= \mu^*(C', T') + \delta_1 + \delta_2 \\
&\leq \mu(C, T) - 3 + p + \delta_1 + \delta_2
\end{aligned}$$

If $p = 0$, then $\mu(C', T') \leq \mu(C, T) - 2$. If $p = 2$, then D is a (semi)cut disc and so $\delta_1 + \delta_2 = 0$. Consequently, in either case, $\mu(C', T') = \mu(C, T) \leq \mu(C, T) - 1$. Since $\mu(C, T) = \mu(C_1, T_1) + \mu(C_2, T_2)$, and since $\mu(C_2, T_2) \geq 0$ by definition, we have $\mu(C_1, T_1) \leq \mu(C', T') < \mu(C, T)$, as desired. □

The next lemma is similar, but will be useful for showing that oriented complexity decreases under some of our simplifying operations.

Lemma 6.7. *Suppose that (C, T) and (C', T') are v.p.-compressionbodies such that (C, T) is not a trivial v.p.-compressionbody, neither T nor T' have vertices, and such that the following hold:*

$$\begin{aligned}
g(\partial_+ C') &= g(\partial_+ C) - 1 \\
|\partial_+ C' \cap T| &\leq |\partial_+ C \cap T| + 2 \\
|\partial_- C'| &= |\partial_- C|
\end{aligned}$$

Then $\mu(C', T') < \mu(C, T)$.

Proof. If (C', T') is a trivial v.p.-compressionbody, $\mu(C', T') = 0 < \mu(C, T)$ by Lemma 6.4. Assume, therefore, that $\mu(C', T') = \mu^*(C', T')$. Then

$$\begin{aligned}\mu(C', T') &\leq 4(g(\partial_+ C) - g(\partial_- C)) - 4 + |\partial_+ C \cap T| + 2 - |\partial_- C \cap T| + 3|\partial_- C| - 3 \\ &= \mu^*(C, T) - 2 \\ &= \mu(C, T) - 2.\end{aligned}$$

□

The next lemma establishes a calculation which will be useful for understanding how oriented complexity behaves under consolidation. Figure 13 encapsulates the notation for the lemma.

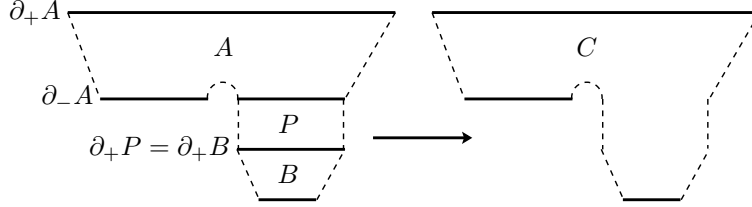


FIGURE 13. The v.p.-compressionbodies A , B , P , and C in Lemma 6.8

Lemma 6.8. *Suppose that (P, T_P) is a trivial product compression body and that (A, T_A) and (B, T_B) are v.p.-compressionbodies with interiors disjoint from each other and from the interior of P . Assume that neither T_A nor T_B have vertices. Assume also that $\partial_- P \subset \partial_- A$ and $\partial_+ B = \partial_+ P$. Let $(C, T) = (A, T_A) \cup (P, T_P) \cup (B, T_B)$. Then*

$$\mu(C, T) = \max(0, \mu(A, T_A) + \mu(B, T_B) + \epsilon),$$

where $\epsilon = -3$ if $(B, T_B) = (B^3, \emptyset)$, $\epsilon = -1$ if $(B, T_B) = (B^3, \text{arc})$, and $\epsilon = 0$ otherwise.

Proof. Since $\partial_- A \neq \emptyset$, the v.p.-compressionbody (A, T_A) is not a trivial ball compressionbody. Hence, $\mu^*(A, T_A) = \mu(A, T_A)$.

If $(B, T_B) = (B^3, \emptyset)$, then (C, T) is obtained by capping off an unpunctured 2-sphere component of $\partial_- A$ with a 3-ball. Hence,

$$\mu^*(C, T) = \mu(A, T_A) - 3 = \mu(A, T_A) + \mu(B, T_B) - 3.$$

If $(B, T_B) = (B^3, \text{arc})$ then (C, T) is obtained by capping off a twice-punctured 2-sphere component of $\partial_- A$ with a 3-ball containing an unknotted arc. Hence,

$$\mu^*(C, T) = \mu(A, T_A) - 1 = \mu(A, T_A) + \mu(B, T_B) - 1.$$

Thus, we may assume that $\mu(B, T_B) = \mu^*(B, T_B)$.

Let $F = \partial_- P$. Observe that $g(F) = g(\partial_+ B)$ and $|F \cap T| = |\partial_+ B \cap T|$. Furthermore,

$$\begin{aligned}\partial_+ C &= \partial_+ A \\ \partial_- C &= (\partial_- A \cup \partial_- B) \setminus F\end{aligned}$$

Hence,

$$\begin{aligned}\mu^*(C, T) &= 4(g(\partial_+ A) - g(\partial_- A) + g(F) - g(\partial_- B)) \\ &\quad + |\partial_+ A \cap T| - |\partial_- A \cap T| + |F \cap T| - |\partial_- B \cap T| \\ &\quad + 3|\partial_- A| + 3|\partial_- B| - 3 \\ &\quad - 3 \\ &= \mu(A, T_A) + \mu(B, T_B).\end{aligned}$$

Since $\mu(A, T_A)$ and $\mu(B, T_B)$ are both non-negative, $\mu^*(C, T) \geq 0$ and so $\mu(C, T) = \mu^*(C, T) = \mu(A, T_A) + \mu(B, T_B)$, as desired. \square

The next corollary follows immediately:

Corollary 6.9. *Suppose that \mathcal{K} is obtained from \mathcal{H} by a consolidation. Then $\mu(\mathcal{K}) \leq \mu(\mathcal{H})$.*

6.3. Oriented complexity.

Definition 6.10. Let \mathbb{S} denote the set of finite sequences in \mathbb{Z} . We place the lexicographic order on \mathbb{S} . That is, if $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_m)$ are two such sequences, we define $x < y$ if and only if one of the following holds:

- there exists $i \leq \min(k, m)$ such that $x_j = y_j$ for all $j < i$ and $x_i < y_i$
- $x_i = y_i$ for all $i \leq \min(k, m)$ and $k < m$.

We say that $x \leq y$ if $x = y$ or if $x < y$. Observe that \leq is a total order on \mathbb{S} and that if $\mathbb{S}' \subset \mathbb{S}$ has the property that there is a universal bound on the absolute value of the terms of the sequences in \mathbb{S}' then \mathbb{S}' is well-ordered by \leq .

In the next definition, we drop the T from the notation for simplicity.

Definition 6.11. Let $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$. Let $f_{\mathcal{H}}: \mathcal{H} \rightarrow \{1, \dots, h\}$ be the height function. Let $n \in \mathbb{N}$ be such that there is a surjective function $f: \mathcal{H}^+ \rightarrow \{1, \dots, n\}$ so that:

- $f(x) = f(y)$ if and only if $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(y)$, and
- If $f_{\mathcal{H}}(x) < f_{\mathcal{H}}(y)$ then $f(x) < f(y)$.

(The function f is the result of restricting $f_{\mathcal{H}}$ to \mathcal{H}^+ and then shifting f values downwards as necessary to make the function surjective.) Let $H_i = f^{-1}(i)$ and recall that $(H_i)_{\downarrow}$ and $(H_i)_{\uparrow}$ are the unions of the v.p.-compressionbodies below or above, respectively, the components of H_i in $M \setminus \mathcal{H}$. Let $\mu_{\downarrow}^i = \mu((H_i)_{\downarrow})$ and $\mu_{\uparrow}^i = \mu((H_i)_{\uparrow})$. Define the **sequential complexity** of \mathcal{H} to be the finite sequence

$$\text{seq } \mathbf{c}(\mathcal{H}) = (\mu_{\downarrow}^1, -\mu_{\uparrow}^1, \mu_{\downarrow}^2, -\mu_{\uparrow}^2, \dots, \mu_{\downarrow}^n, -\mu_{\uparrow}^n).$$

That is, $\text{seq } \mathbf{c}(\mathcal{H})$ is the sequence in \mathbb{S} whose $2k$ th term is the *negative* of the index of $(H_k)_{\uparrow}$ and whose $(2k-1)$ st term is the index of $(H_k)_{\downarrow}$, for all $k \in \mathbb{N}$. The **oriented complexity** of \mathcal{H} is defined to be the pair

$$\overrightarrow{\mathbf{c}}(\mathcal{H}) = (\mu(\mathcal{H}), \text{seq } \mathbf{c}(\mathcal{H})).$$

Oriented complexities are compared lexicographically as in Definition 6.10.

Remark 6.12. Observe that each term of $\text{seq } \mathbf{c}(\mathcal{H})$ is no more than the total index $\mu(\mathcal{H})$ in absolute value, so if $\mathbb{S}' \subset \mathbb{S}$ is a set of sequential complexities for multiple v.p.-bridge surfaces with bounded total index, then \mathbb{S}' is well-ordered.

Lemma 6.13. *Assume that every sphere in M separates and that $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$. Assume that no component of $\mathcal{H}^- \cup \partial M$ is a sphere intersecting T exactly once. Also assume that no component of $(M, T) \setminus \mathcal{H}$ is a trivial product compressionbody adjacent to a component of \mathcal{H}^- . Let $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$ be obtained by an elementary thinning sequence applied to $H \subset \mathcal{H}^+$. Then*

$$\overrightarrow{\mathbf{c}}(\mathcal{K}) < \overrightarrow{\mathbf{c}}(\mathcal{H})$$

Proof. Since the index of a v.p.-compressionbody is calculated by first drilling out vertices, without loss of generality, we may assume that T has no vertices. Let \mathcal{H}_1 , \mathcal{H}_2 , and $\mathcal{H}_3 = \mathcal{K}$ be the multiple v.p.-bridge surfaces from the definition of elementary thinning sequence (Definition 5.7). Let D_- and D_+ be the sc-discs used to untelescope H , with D_- lying below H . Let H_\pm be the components of \mathcal{H}_2^+ obtained by compressing H along D_\pm and, if D_\pm is separating, discarding a component of the compressed surface. Passing from \mathcal{H}_2 to \mathcal{H}_3 we consolidate H_- or H_+ or both with components of $\mathcal{H}^- \subset \mathcal{H}_2^-$, if such a consolidation is possible.

Let $F \subset \mathcal{H}_1^- \cap \mathcal{H}_2^- \cap \mathcal{H}_3^-$ be the new thin surface of $\mathcal{K}^- \setminus \mathcal{H}^-$. Since $F \neq \emptyset$ is obtained by an sc-compression either of H_- or H_+ , the v.p.-compressionbodies $(H_-)_\uparrow$ and $(H_+)_\downarrow$ are non-trivial.

The v.p.-compressionbody $(H_-)_\downarrow$ in $M \setminus \mathcal{H}_2$ is obtained by reducing H_\downarrow using the sc-disc D_- and discarding a component, if D_- was separating. Thus, by Corollary 6.6, $\mu(H_-)_\downarrow < \mu(H_\downarrow)$. In examining the effect on sequential continuity, there are two cases to consider: when H is the unique thick surface of \mathcal{H}^+ at its height and when it is not the unique thick surface of \mathcal{H}^+ at its height. We consider only the first possibility and leave the second as a nearly identical exercise for the reader. In what follows, we often suppress T from the notation, writing $\mu(H_\downarrow)$ for $\mu(H_\downarrow, T \cap H_\downarrow)$ for example.

The sequential complexity of \mathcal{H} is

$$\text{seq } \mathbf{c}(\mathcal{H}) = (\cdots, \mu(H_\downarrow), -\mu(H_\uparrow), \cdots)$$

Supposing that $f(H) = i$, the term $\mu(H_\downarrow)$ is in the $(2i - 1)$ st term of the sequence.

$$\text{seq } \mathbf{c}(\mathcal{H}_2) = (\cdots, \mu((H_-)_\downarrow), -\mu((H_-)_\uparrow), \mu((H_+)_\downarrow), -\mu((H_+)_\uparrow), \cdots)$$

with $\mu((H_-)_\downarrow)$ in the $(2i - 1)$ term. All the terms of $\text{seq } \mathbf{c}(\mathcal{H}_2)$ prior to the $(2i - 1)$ st term are the same as for $\text{seq } \mathbf{c}(\mathcal{H})$. Thus, $\text{seq } \mathbf{c}(\mathcal{H}_2) < \text{seq } \mathbf{c}(\mathcal{H})$.

If, when passing from \mathcal{H}_2 to \mathcal{H}_3 , the surface H_- is not consolidated with a surface in \mathcal{H}^- , then the first $(2i - 1)$ terms of $\text{seq } \mathbf{c}(\mathcal{H}_3)$ coincide with those of $\text{seq } \mathbf{c}(\mathcal{H}_2)$ and so $\text{seq } \mathbf{c}(\mathcal{K}) = \text{seq } \mathbf{c}(\mathcal{H}_3) < \text{seq } \mathbf{c}(\mathcal{H})$ as desired.

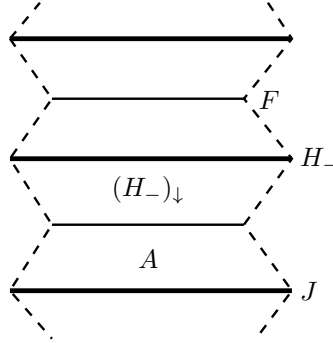


FIGURE 14. The surfaces appearing in the creation of \mathcal{H}_3 from \mathcal{H}_2 .

Suppose, therefore, that in passing from \mathcal{H}_2 to \mathcal{H}_3 , we consolidate H_- with a surface in \mathcal{H}^- . This happens only if $(H_-)_\downarrow$ is a trivial product compressionbody with $\partial_-(H_-)_\downarrow \subset \mathcal{H}^-$. Let (A, T_A) be the component of $M \setminus \mathcal{H}$ containing $\partial_-(H_-)_\downarrow$ and let $J = \partial_+ A$. Figure 14 gives a schematic depiction of the set-up. Let $j = f(J)$. After the consolidation, we have $J_\uparrow = A \cup (H_-)_\downarrow \cup (H_-)_\uparrow$. By Lemma 6.8, for the compressionbody J_\uparrow in $M \setminus \mathcal{H}_3$, we have:

$$\mu(J_\uparrow) = \mu(A) + \mu(H_-)_\uparrow.$$

The sequential complexity of \mathcal{H}_2 is:

$$\text{seq } \mathbf{c}(\mathcal{H}_2) = (\cdots, x + \mu(J_\downarrow), -y - \mu(A), \mu((H_-)_\downarrow), -\mu((H_-)_\uparrow), \mu((H_+)_\downarrow), -\mu((H_+)_\uparrow), \cdots)$$

where x and y are non-negative integers corresponding to compressionbodies adjacent to thick surfaces of \mathcal{H}_2^+ at the same height as J . The term $x + \mu(J_\downarrow)$ is in the $(2i - 3)$ rd spot of the sequence.

After the consolidation, we see that the sequential complexity of $\mathcal{K} = \mathcal{H}_3$ is

$$\text{seq } \mathbf{c}(\mathcal{K}) = (\cdots, x + \mu(J_\downarrow), -y - \mu(A) - \mu(H_-)_\uparrow, \cdots)$$

where the first through $(2i - 3)$ rd terms are the same as for $\text{seq } \mathbf{c}(\mathcal{H}_2)$. Since $(H_-)_\uparrow$ is not a trivial v.p.-compressionbody, by Lemma 6.4, $-\mu((H_-)_\uparrow) < 0$. Hence,

$$\text{seq } \mathbf{c}(\mathcal{K}) < \text{seq } \mathbf{c}(\mathcal{H}_2) < \text{seq } \mathbf{c}(\mathcal{H}),$$

as desired.

Finally, we confirm that $\mu(\mathcal{K}) \leq \mu(\mathcal{H})$.

Consider, first, \mathcal{H}_2 . We must show that the sum of the indices of $(H_-)_\downarrow$, $(H_-)_\uparrow$, $(H_+)_\downarrow$, and $(H_+)_\uparrow$ is no more than the sum of the indices of H_\downarrow and H_\uparrow .

We have:

$$\begin{aligned} & \mu^*((H_-)_\downarrow) + \mu^*((H_-)_\uparrow) + \mu^*((H_+)_\downarrow) + \mu^*((H_+)_\uparrow) = \\ & 8g(H_-) + 8g(H_+) - 8g(F') - 4g(\partial_- H_\downarrow) - 4g(\partial_- H_\uparrow) \\ & + 2|H_- \cap T| + 2|H_+ \cap T| - 2|F \cap T| - |\partial_- H_\downarrow \cap T| + |\partial_- H_\uparrow \cap T| \\ & 3|\partial_- H_\downarrow| + 3|\partial_- H_\uparrow| + 3|F| - 12 \end{aligned}$$

Recall that $g(H_-) + g(H_+) - g(F) = g(H)$ and $|H_- \cap T| + |H_+ \cap T| - |F \cap T| = |H \cap T|$. Then:

$$\begin{aligned} & \mu^*((H_-)_\downarrow) + \mu^*((H_-)_\uparrow) + \mu^*((H_+)_\downarrow) + \mu^*((H_+)_\uparrow) = \\ & 8g(H) - 4g(\partial_- H_\downarrow) - 4g(\partial_- H_\uparrow) \\ & + 2|H \cap T| - |\partial_- H_\downarrow \cap T| - |\partial_- H_\uparrow \cap T| \\ & 3|\partial_- H_\downarrow| + 3|\partial_- H_\uparrow| - 3 - 3 + 3|F| - 6 = \\ & \mu^*(H_\downarrow) + \mu^*(H_\uparrow) + 3|F| - 6 \leq \\ & \mu(H_\downarrow) + \mu(H_\uparrow) + 3|F| - 6. \end{aligned}$$

Since $|F| \leq 2$, this give us

$$\mu^*((H_-)_\downarrow) + \mu^*((H_-)_\uparrow) + \mu^*((H_+)_\downarrow) + \mu^*((H_+)_\uparrow) \leq \mu(H_\downarrow) + \mu(H_\uparrow),$$

with equality if and only if $|F| = 2$.

Neither $(H_-)_\uparrow$ nor $(H_+)_\downarrow$ is a trivial v.p.-compressionbody, so $\mu^*((H_-)_\uparrow) = \mu((H_-)_\uparrow)$ and $\mu^*((H_+)_\downarrow) = \mu((H_+)_\downarrow)$. Let $\delta_- = \mu((H_-)_\downarrow) - \mu^*((H_-)_\downarrow)$ and $\delta_+ = \mu((H_+)_\uparrow) - \mu^*((H_+)_\uparrow)$. Since D_- and D_+ were sc-discs, by Lemma 6.4, neither $(H_-)_\downarrow$ nor $(H_+)_\uparrow$ is (B^3, \emptyset) . Hence, by Lemma 6.4, $\delta_\pm \in \{0, 1\}$.

Thus,

$$\begin{aligned} & \mu((H_-)_\downarrow) + \mu((H_-)_\uparrow) + \mu((H_+)_\downarrow) + \mu((H_+)_\uparrow) = \\ & \mu^*((H_-)_\downarrow) + \mu^*((H_-)_\uparrow) + \mu^*((H_+)_\downarrow) + \mu^*((H_+)_\uparrow) + \delta_- + \delta_+ \leq \\ & \mu(H_\downarrow) + \mu(H_\uparrow) + \delta_- + \delta_+ + 3|F| - 6. \end{aligned}$$

If $|F| = 1$ or if $\delta_- + \delta_+ = 0$, then since $\delta_- + \delta_+ \leq 2$, we have

$$\mu((H_-)_\downarrow) + \mu((H_-)_\uparrow) + \mu((H_+)_\downarrow) + \mu((H_+)_\uparrow) < \mu(H_\downarrow) + \mu(H_\uparrow).$$

This implies that $\mu(\mathcal{H}_2) \leq \mu(\mathcal{H})$.

If $|F| = 2$, then both D_- and D_+ are non-separating, but $\partial D_- \cup \partial D_+$ is separating on H . This implies that no consolidation occurs when passing from \mathcal{H}_1 to \mathcal{H}_2 . In particular, $g(H_-) = g(H_+) =$

$g(H) - 1$. If $\delta_{\pm} = 1$, then $(H_-)_{\downarrow}$ or $(H_+)_{\uparrow}$ is a trivial ball compression body. In which case, $g(H_{\pm}) = 0$ and so $g(H) = 1$. Since ∂D_- and ∂D_+ are disjoint, they are parallel on the torus H (ignoring $T \cap H$). Hence, M contains a non-separating sphere, contradicting our hypotheses.

Therefore, $\mu(\mathcal{H}_2) \leq \mu(\mathcal{H})$. When passing from \mathcal{H}_2 to $\mathcal{H}_3 = \mathcal{K}$ we consolidate H_- , H_+ , or both. By Lemma 6.8, we see that the total index of \mathcal{K} is at most the total index of \mathcal{H}_2 . Hence, $\mu(\mathcal{K}) \leq \mu(\mathcal{H})$. \square

Finally, we need to confirm that oriented complexity decreases under destabilization, unperturbing, and undoing a removable arc. We do this by showing that total index strictly decreases under those operations.

Lemma 6.14. *Assume that no component of ∂M is a sphere intersecting T two or fewer times. Suppose that $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ such that no component of \mathcal{H}^- is a sphere intersecting T exactly once. Assume that \mathcal{K} is obtained from \mathcal{H} by a destabilization, unperturbing, or undoing a removable arc. Then $\mu(\mathcal{K}) < \mu(\mathcal{H})$ and $\overrightarrow{\mathcal{C}}(\mathcal{K}) < \overrightarrow{\mathcal{C}}(\mathcal{H})$.*

Proof. It suffices to show $\mu(\mathcal{K}) < \mu(\mathcal{H})$.

Let $H \subset \mathcal{H}^+$ be the component with the generalized stabilization, perturbation, or removable arc that we eliminate when we create \mathcal{K} . Let $H' \subset \mathcal{K}^+$ be the new thick surface. Without loss of generality, we may assume that T has no vertices.

Case 1: H has a stabilization or meridional stabilization.

Without loss of generality, assume that we compress along a c-disc $D \subset H_{\downarrow}$ to create H' from H . Then (H'_{\downarrow}) is obtained from (H_{\downarrow}) by reducing along a c-disc. Hence, by Corollary 6.6, $\mu(H'_{\downarrow}) < \mu(H_{\downarrow})$. On the other hand, by Lemma 6.7, we have also have $\mu(H'_{\uparrow}) < \mu(H_{\uparrow})$. Thus, $\mu(\mathcal{K}) < \mu(\mathcal{H})$.

Case 2: H contains a ∂ -stabilization, a meridional ∂ -stabilization, a ghost ∂ -stabilization, or a meridional ghost ∂ -stabilization.

Suppose that H is (meridionally) (ghost) boundary-stabilized along a connected graph G with edges e_1, \dots, e_m . Let R_1, \dots, R_n be the components of ∂M that e_1, \dots, e_m are adjacent to and let S be the boundary component of a regular neighborhood of the graph G and the surfaces R_1, \dots, R_n that lies in H_{\downarrow} . The surface H is (meridionally) boundary-stabilized along S . If H is simply (meridionally) ∂ -stabilized, we take $m = 0$ and $n = 1$. Since S is connected, $m + 1 \geq n > 0$.

Let D be the separating sc-disc defining the generalized stabilization. Compressing H along D creates a disconnected surface having H' as one component and having its other component H'' be parallel to S . Let $p = |D \cap T|$ and recall that $p \in \{0, 1\}$.

As before, we may assume that $D \subset H_{\uparrow}$, so that $\mu(H'_{\uparrow}, T'_{\uparrow}) < \mu(H_{\uparrow}, T_{\uparrow})$ by Corollary 6.6. We will show that $\mu(H'_{\downarrow}, T'_{\downarrow}) \leq \mu(H_{\downarrow}, T_{\downarrow})$. It will then be immediate that $\mu(\mathcal{K}) < \mu(\mathcal{H})$ and $\overrightarrow{\mathcal{C}}(\mathcal{K}) < \overrightarrow{\mathcal{C}}(\mathcal{H})$.

If $\mu^*(H'_{\downarrow}, T'_{\downarrow}) < \mu(H'_{\downarrow}, T'_{\downarrow})$, we have $\mu(H'_{\downarrow}, T'_{\downarrow}) = 0 \leq \mu(H_{\downarrow}, T_{\downarrow})$, as desired. Assume, therefore, that $\mu(H'_{\downarrow}, T'_{\downarrow}) = \mu^*(H'_{\downarrow}, T'_{\downarrow})$. Observe the following, reasons for each are given subsequently.

$$\begin{aligned} 4(g(H) - g(H')) &= 4g(H'') & (1) \\ 4(-g(\partial_- H_{\downarrow}) + g(\partial_- H'_{\downarrow})) &= +4(g(R_1) + \dots + g(R_n)) & (2) \\ |H \cap T| - |H' \cap T| &= |H'' \cap T| - 2p & (3) \\ -|\partial_- H_{\downarrow} \cap T| + |\partial_- H'_{\downarrow} \cap T| &= |R_1 \cap T| + \dots + |R_n \cap T| & (4) \\ 3(|\partial_- H_{\downarrow}| - |\partial_- H'_{\downarrow}|) &= -3n & (5) \end{aligned}$$

Here are the reasons for each of the preceding equalities:

- (1) We discard H'' to create H' from H .
- (2) We move R_1, \dots, R_n from $\partial_- H_\uparrow$ to $\partial_- H'_\downarrow$.
- (3) $|H' \cap T| + |H'' \cap T| = |H \cap T| + 2p$.
- (4) We move R_1, \dots, R_n from $\partial_- H_\uparrow$ to $\partial_- H'_\downarrow$.
- (5) We move R_1, \dots, R_n from $\partial_- H_\uparrow$ to $\partial_- H'_\downarrow$.

Thus,

$$\begin{aligned} \mu(H_\downarrow, T_\downarrow) - \mu(H'_\downarrow, T'_\downarrow) &= \mu^*(H_\downarrow, T_\downarrow) - \mu^*(H'_\downarrow, T'_\downarrow) \\ &= 4g(H'') + 4(g(R_1) + \dots + g(R_n)) + |H'' \cap T| - 2p \\ &\quad + |R_1 \cap T| + \dots + |R_n \cap T| - 3n. \end{aligned}$$

Let m_i be the number of endpoints of $e_1 \cup \dots \cup e_m$ on R_i . Note that

$$\begin{aligned} 2m &= m_1 + \dots + m_n \\ g(H'') &= g(R_1) + \dots + g(R_n) + m - n + 1 \\ |H'' \cap T| &= |R_1 \cap T| + \dots + |R_n \cap T| - (m_1 + \dots + m_n) \end{aligned}.$$

Thus,

$$\begin{aligned} &\mu(H_\downarrow, T_\downarrow) - \mu(H'_\downarrow, T'_\downarrow) \\ &\geq 8 \sum_{i=1}^n g(R_i) + 4(m - n + 1) + 2 \sum_{i=1}^n (|R_i \cap T| - m_i) - 2p - 3n \\ &= \sum_{i=1}^n (8g(R_i) + m_i + 2|R_i \cap T| - 7) + 4 - 2p \\ &\geq \sum_{i=1}^n (8g(R_i) + m_i + 2|R_i \cap T| - 7) + 2. \end{aligned}$$

If, for some m_i , we have $m_i = 0$, then, $G = \emptyset$, and $m = 0$ and $n = 1$. From the above calculations we see that

$$\mu(H_\downarrow, T_\downarrow) - \mu(H'_\downarrow, T'_\downarrow) \geq 8g(R_1) + 2|R_1 \cap T| - 5.$$

If $g(R_1) = 0$, then by assumption $|R_1 \cap T| \geq 3$. Hence, no matter what the genus of R_1 is, we have

$$\mu(H_\downarrow, T_\downarrow) - \mu(H'_\downarrow, T'_\downarrow) \geq 1,$$

as desired.

Assume, therefore, that for all i , $m_i \geq 1$. Hence,

$$\mu(H_\downarrow, T_\downarrow) - \mu(H'_\downarrow, T'_\downarrow) = \sum_{i=1}^n (8g(R_i) + m_i + 2|R_i \cap T| - 7) + 2$$

If $g(R_i) \geq 1$, then the summand $(8g(R_i) + m_i + 2|R_i \cap T| - 7) \geq 1$. If $g(R_i) = 0$, then $|R_i \cap T| \geq 3$. In which case, the summand $(m_i + 2|R_i \cap T| - 7) \geq 0$. Thus, $\mu(H_\downarrow, T_\downarrow) - \mu(H'_\downarrow, T'_\downarrow) > 0$, as desired.

Case 4: H is perturbed or removable.

In both of these cases, we create \mathcal{K} from \mathcal{H} by isotoping H to H' in such a way that $|H' \cap T| = |H \cap T| - 2$. The effect on $\mu(H_\downarrow)$ and $\mu(H_\uparrow)$ is to decrease them by 2, since neither can be a trivial v.p.-compressionbody and all other quantities in the definition of μ^* remain the same. Hence, $\mu(\mathcal{K}) < \mu(\mathcal{H})$. \square

7. THINNING SEQUENCES, THIN POSITION

In this section, we use our collection of simplifying moves to put a partial order on a certain subset of $\overrightarrow{vp\mathbb{H}}(M, T)$. The least elements of the partial order will be called “locally thin” and will have a number of very useful properties.

Definition 7.1. An oriented v.p.-multiple bridge surface \mathcal{H} is **reduced** if it does not contain a generalized stabilization, a perturbation, or a removable arc and if no component of $(M, T) \setminus \mathcal{H}$ is a trivial product compressionbody adjacent to a component of \mathcal{H}^- .

Definition 7.2. Suppose that $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ is reduced and that T is irreducible. An **extended thinning move** applied to \mathcal{H} consists of the following steps in the following order:

- (1) Perform an elementary thinning sequence
- (2) Destabilize, unperturb, and undo removable arcs until no generalized stabilizations, perturbations, or removable arcs remain
- (3) Consolidate all components of \mathcal{H}^- and \mathcal{H}^+ cobounding a trivial product compressionbody in $(M, T) \setminus \mathcal{H}$
- (4) Repeat (2) and (3) as much as necessary until \mathcal{H} does not have a generalized stabilization, perturbation, or removable arc or product region adjacent to \mathcal{H}^- .

For $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$, write $\mathcal{H} \rightarrow \mathcal{K}$, and say that \mathcal{H} **thins to** \mathcal{K} or that \mathcal{K} is obtained by **thinning** \mathcal{H} , if there is a (possibly empty) sequence of extended thinning moves producing \mathcal{K} from \mathcal{H} .

Remark 7.3. Note that by Lemma 6.14 and the fact that \mathcal{H} has only finitely many components each intersecting T only finitely many times, steps (2) and (3) cannot be applied arbitrarily many times, so an extended thinning move is well-defined and results in a reduced oriented multiple v.p.-bridge surface. Furthermore, if \mathcal{H}' is the result of performing an elementary thinning sequence to \mathcal{H} , then by Lemma 5.9, no component of $(M, T) \setminus \mathcal{H}'$ is a trivial product v.p.-compressionbody bounded by a component of \mathcal{H}'^+ and a component of \mathcal{H}'^- . Thus, we never begin applying Step (3) without having first (non-trivially) applied Step (2).

Recall that in a poset, a “least element” is an element x with the property that no element is strictly less than x . In our context, we say that an element $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$ is a **least element** or **locally thin** if it is reduced and if $\mathcal{K} \rightarrow \mathcal{K}'$ implies that $\mathcal{K} = \mathcal{K}'$.

Theorem 7.4. *Let (M, T) be a (3-manifold, graph) pair. Suppose that no component of ∂M is a sphere intersecting T two or fewer times and that every sphere in M separates. Then the relation \rightarrow is a partial order on the set of reduced elements of $\overrightarrow{vp\mathbb{H}}(M, T)$. Furthermore, if $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ is reduced, then there is a least element (i.e. locally thin) $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$ such that $\mathcal{H} \rightarrow \mathcal{K}$. If \mathcal{H} is linear, then so is \mathcal{K} .*

Proof. We have already observed that \rightarrow is a relation on the set of reduced multiple v.p.-bridge surfaces for (M, T) . Since each reduced \mathcal{H} can be obtained from itself by an empty sequence of extended thinning moves, \rightarrow is reflexive. By definition, \rightarrow is transitive.

Suppose that \mathcal{H} is reduced and that \mathcal{K} is obtained from \mathcal{H} by a single extended thinning move. By Lemma 6.13, Step (1) of an extended thinning move strictly decreases oriented complexity. By Lemma 6.14, a non-trivial application of Step (2) strictly decreases total index and thus also strictly decreases oriented complexity. By Corollary 6.9, Step (3) does not increase total index. Hence, $\overrightarrow{c}(\mathcal{K}) < \overrightarrow{c}(\mathcal{H})$. Thus, \rightarrow is anti-symmetric. Since \rightarrow is reflexive, anti-symmetric, and transitive, it is a partial order.

Remark 6.12 observes that each term of $\text{seq } \mathbf{c}(\mathcal{K})$ is bounded above in absolute value by $\mu(\mathcal{K}) \leq \mu(\mathcal{H})$. Hence, as observed in that remark, the set of oriented complexities for multiple v.p.-bridge surfaces \mathcal{J} with $\vec{\mathbf{c}}(\mathcal{J}) \leq \vec{\mathbf{c}}(\mathcal{H})$ is well-ordered. Thus, we cannot apply an infinite sequence of extended thinning moves to \mathcal{H} . Consequently, there exists a least element (i.e. locally thin) $\mathcal{K} \in \overrightarrow{vp\mathbb{H}}(M, T)$ with $\mathcal{H} \rightarrow \mathcal{K}$.

By Lemma 5.10, elementary thinning sequences preserve linearity. It is evident from the definitions that Steps (2) and (3) of an extended thinning move also preserve linearity (See Remark 4.2 and Lemmas 4.5, 4.7, and 5.6). Hence, if \mathcal{H} is linear, then so is \mathcal{K} . \square

8. SWEEPOUTS

Sweepouts, as in most applications of thin position, are the key tool for finding disjoint compressing discs on two sides of a thick surface. In this section, we will use $X - Y$ to denote the set-theoretic complement of Y in X , as opposed to $X \setminus Y$ which indicates the complement of an open regular neighborhood of Y in X .

Definition 8.1. Suppose that (C, T) is a v.p.-compressionbody and that $\Sigma \subset C$ is a trivalent graph embedded in C such that the following hold:

- $(C, T) \setminus \Sigma$ is homeomorphic to $(\Sigma \times I, \text{vertical arcs})$
- Σ contains the ghost arcs of T and no interior vertex of Σ lies on a ghost arc
- Each boundary vertex of Σ lies on T or on $\partial_- C$.
- Any edge of T which is not a ghost arc and which intersects Σ is a bridge arc intersecting Σ in a boundary vertex.

Then Σ is a spine for (C, T) . See Figure 15 for an example.

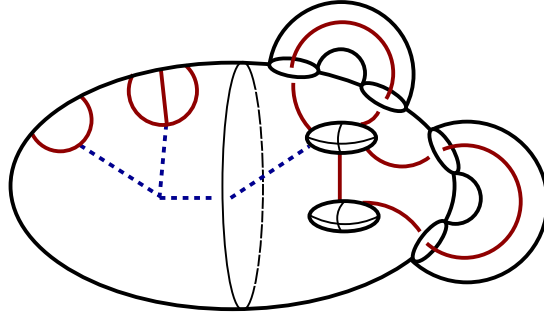


FIGURE 15. A spine for the v.p.-compressionbody from Figure 3 consists of the dashed blue graph together with the edges of T that are disjoint from $\partial_+ C$.

Suppose that $H \in vp\mathbb{H}(M, T)$ is connected and that Σ_\uparrow and Σ_\downarrow are spines for $(H_\uparrow, T \cap H_\uparrow)$ and $(H_\downarrow, T \cap H_\downarrow)$. The manifold $M - (\Sigma_\uparrow \cup \Sigma_\downarrow)$ is homeomorphic to $H \times (0, 1)$ by a map taking $T - (\Sigma_\uparrow \cup \Sigma_\downarrow)$ to vertical edges. We may extend the homeomorphism to a map $h: M \rightarrow I$ taking Σ_\downarrow to -1 and Σ_\uparrow to $+1$. The map h is called a **sweepout** of M by H . Note that for each $t \in (0, 1)$, $H_t = h^{-1}(t)$ is properly isotopic in $M \setminus T$ to $H \setminus T$, that $h^{-1}(-1) = \partial_- H_\downarrow \cup \Sigma_\downarrow$, and $h^{-1}(1) = \partial_- H_\uparrow \cup \Sigma_\uparrow$. If we perturb h by a small isotopy, we also refer to the resulting map as a sweepout.

Theorem 8.2. *Let (M, T) be a $(3\text{-manifold, graph})$ pair. Suppose that $F \subset (M, T)$ and assume that $H \in \text{vp}\mathbb{H}(M, T)$ is connected and doesn't bound a trivial v.p.-compressionbody on either side. Then, H can be isotoped transversally to T such that after the isotopy H and F are transverse and one of the following holds*

- (1) $H \cap F = \emptyset$
- (2) $H \cap F \neq \emptyset$, every component of $H \cap F$ is essential in F and no component of $H \cap F$ bounds an sc-disc for H .
- (3) H is sc-weakly reducible.

Remark 8.3. The essence of this argument can be found in many places. It originates with Gabai's original thin position argument [3] and is adapted to the context of Heegaard splittings by Rubinstein and Scharlemann [10]. A version for graphs in S^3 plays a central role in [4].

Proof. Let h be a sweepout corresponding to H , as above. Perturb the map h slightly so that $h|_F$ is Morse with critical points at distinct heights. Let

$$0 = v_0 < v_1 < v_2 < \cdots < v_n = 1$$

be the critical values of $h|_F$. Let $I_i = (v_{i-1}, v_i)$. Label I_i with \downarrow (resp. \uparrow) if some component of $F \cap H_t$ bounds a sc-disc below (resp. above) H_t for some $t \in I_i$.

Observe that, by standard Morse theory, the label(s) on I_i are independent of the choice of $t \in I_i$.

Case 1: Some interval I_i is without a label.

Let $t \in I_i$. If $H_t \cap F = \emptyset$, then we are done, so suppose that $H_t \cap F \neq \emptyset$.

Suppose that some component $\zeta \subset F$ is inessential in F . Without loss of generality, we may assume that ζ is innermost in F . Let $D \subset F$ be the disc or once-punctured disc it bounds. Since ζ does not bound an sc-disc for H_t , the disc D is properly isotopic in $M \setminus T$, relative to ∂D into H_t . Let $B \subset M$ be the 3-ball bounded by D and the disc in H_t . By an isotopy supported in a regular neighborhood of B , we may isotope H_t to eliminate ζ (and possibly some other inessential curves of $H_t \cap F$). Repeating this type of isotopy as many times as necessary, we may assume that no curve of $H_t \cap F$ is inessential in F . If $H_t \cap F = \emptyset$, we have the first conclusion. If $H_t \cap F \neq \emptyset$, then we have the Conclusion (2).

Suppose, therefore, that each I_i has a label.

Case 2: Some I_i is labelled both \uparrow and \downarrow .

Since for each $t \in I_i$, H_t is transverse to F we have Conclusion (3).

Case 3: There is an i so that I_i is labelled \downarrow and I_{i+1} is labelled \uparrow , or vice versa.

The labels cannot change from I_i to I_{i+1} at any tangency other than a saddle tangency. Let $\epsilon > 0$ be smaller than the lengths of the intervals I_i and I_{i+1} . Since H_t is orientable, under the projections of $H_{v_i-\epsilon}$ and $H_{v_i+\epsilon}$ to H , the 1-manifold $H_{v_i-\epsilon} \cap F$ can be isotoped to be disjoint from $H_{v_i+\epsilon} \cap F$. Since some component of the former set bounds an sc-disc on one side of H and some component of the latter set bounds an sc-disc on the other side of H , we have Conclusion 3 again.

Case 4: For every i , I_i is labelled \downarrow and not \uparrow or for every i , I_i is labelled \uparrow and not \downarrow .

Without loss of generality, assume that each I_i is labelled \uparrow and not \downarrow . In particular, I_1 is labelled \uparrow and not \downarrow . Fix $t \in I_1$ and consider H_t . Since H does not bound a trivial v.p.-compressionbody to either side, the spine for $(H_\downarrow, T \cap H_\downarrow)$ has an edge e . Since I_i is below the lowest critical point for $h|_F$, the components of $F \cap (H_t)_\downarrow$ intersecting e are a regular neighborhood in F of $F \cap e$. Let

D_\downarrow be a meridian disc for e with boundary in H_t and which is disjoint from $F \cap (H_t)_\downarrow$. Since I_1 is labelled \uparrow , there is a component $\zeta \subset H_t \cap F$ such that ζ bounds an sc-disc D_\uparrow for H_t in $(H_t)_\uparrow$. The pair $\{D_\uparrow, D_\downarrow\}$ is then a weak reducing pair for H_t , giving Conclusion (3). \square

Remark 8.4. Observe that in Conclusion (3), we can only conclude that H is sc-weakly reducible – not that H is c-weakly reducible. This arises in Case 4 of the proof, when we use an edge of the spine to produce an sc-disc. This is one reason for allowing semi-compressing and semi-cut discs in weak reducing pairs.

Corollary 8.5. *Suppose that $H \in \text{vp}\mathbb{H}(M, T)$ is connected and sc-strongly irreducible. If a component S of ∂M is c-compressible, then the component of $(M, T) \setminus H$ containing S is a trivial product compressionbody.*

Proof. Let $F \subset (M, T)$ be a c-disc for S . Let (C, T_C) and (E, T_E) be the components of $(M, T) \setminus H$, with $S \subset \partial_- C$. If (E, T_E) is a trivial compressionbody, we may isotope F out of E to be contained in C . This contradicts the fact that $\partial_- C$ is c-incompressible in C . Hence, (E, T_E) is not a trivial product compressionbody. Since $S \subset \partial_- C$ is c-compressible, it either has positive genus or intersects T at least 3 times. In particular, (C, T_C) is not a trivial ball compressionbody. Suppose, for a contradiction, that (C, T_C) is not a trivial product compressionbody. Then by Theorem 8.2 H can be isotoped transversally to T such that after the isotopy one of the following holds:

- (1) $H \cap F = \emptyset$
- (2) $H \cap F \neq \emptyset$, every component of $H \cap F$ is essential in F and no component of $H \cap F$ bounds an sc-disc for H .

Since $\partial_- C$ is c-incompressible in C , by Lemma 3.4, the first conclusion cannot hold. Since no curve in a disc or once-punctured disc is essential, the second conclusion is also impossible. Thus, (C, T_C) is a trivial product compressionbody. \square

Theorem 8.6 (Properties of locally thin surfaces). *Suppose that (M, T) is a (3-manifold, graph) pair, with T irreducible. Let $\mathcal{H} \in \overrightarrow{\text{vp}\mathbb{H}}(M, T)$ be locally thin. Then the following hold:*

- (1) \mathcal{H} is reduced
- (2) Each component of \mathcal{H}^+ is sc-strongly irreducible in the complement of \mathcal{H}^- .
- (3) No component of $(M, T) \setminus \mathcal{H}$ is a trivial product compressionbody between \mathcal{H}^- and \mathcal{H}^+ .
- (4) Every component of \mathcal{H}^- is c-essential in (M, T) .
- (5) If (M, T) is irreducible and if \mathcal{H} contains a 2-sphere disjoint from T , then $T = \emptyset$ and $M = S^3$ or $M = B^3$.

Proof. Without loss of generality, we may assume that T has no vertices (drilling them out to turn them into components of ∂M if necessary). Conclusions (1) and (3) are immediate from the definition of locally thin. If some component of \mathcal{H}^+ is sc-weakly reducible in $(M, T) \setminus \mathcal{H}^-$, then, since T is irreducible, we could perform an elementary thinning sequence, contradicting the definition of locally thin. Thus, (2) also holds.

Next we show that each component of \mathcal{H}^- is c-incompressible. Suppose, therefore, that $S \subset \mathcal{H}^-$ is a thin surface. We first show that S is c-incompressible and then that it is not ∂ -parallel. Suppose that S is c-compressible by a c-disc D . By an innermost disc argument, we may assume that no curve of $D \cap (\mathcal{H}^- \setminus S)$ is an essential curve in \mathcal{H}^- . By passing to an innermost disc, we may also assume that $D \cap (\mathcal{H}^- \setminus S) = \emptyset$. Let (M_0, T_0) be the component of $(M, T) \setminus \mathcal{H}^-$ containing D . Let

$H = \mathcal{H}^+ \cap M_0$ and recall that H is connected. By Corollary 8.5 applied to H in (M_0, T_0) , the v.p.-compressionbody between S and H is a trivial product compressionbody. This contradicts property (3) of locally thin multiple v.p.-bridge surfaces. Thus, each component of \mathcal{H}^- is c-incompressible.

We now show no sphere component of \mathcal{H}^- bounds a 3-ball in $M \setminus T$. Suppose that $S \subset \mathcal{H}^-$ is such a sphere and let $B \subset M \setminus T$ be the 3-ball it bounds. By passing to an innermost such sphere, we may assume that no component of \mathcal{H}^- in the interior of B is a 2-sphere. If there is a component of \mathcal{H}^- in the interior of B , that component would be compressible, a contradiction. Thus the intersection H of \mathcal{H} with the interior of B is a component of \mathcal{H}^+ . The surface H is a Heegaard splitting of B . If H is a sphere it is parallel to S , contradicting (3). If H is not a sphere, then by [20] it is stabilized, contradicting (1). Thus, each component of \mathcal{H}^- is c-incompressible in (M, T) and not a sphere bounding a 3-ball in $M \setminus T$. In particular, if (M, T) is irreducible no component of \mathcal{H}^- is a sphere disjoint from T .

We now show that no component of \mathcal{H}^- is ∂ -parallel. Suppose, to obtain a contradiction, that a component F of \mathcal{H}^- is boundary parallel in the exterior of T . An analysis (which we provide momentarily) of the proof of [18, Theorem 9.3] shows that \mathcal{H} either has a perturbation or a generalized stabilization or is removable. We elaborate on this:

As in [18, Lemma 3.3], since all components of \mathcal{H}^- are c-incompressible, we may assume that the product region W between F and $\partial(M \setminus T)$ has interior disjoint from \mathcal{H}^- . (That is, F is innermost.) Observe that W is a compressionbody with $F = \partial_+ W$ and the component $H = \mathcal{H}^+ \cap W$ is a v.p.-bridge surface for $(W, T \cap W)$.

If there is a component of $T \cap W$ with both endpoints on F , then F must be a 2-sphere and W is a 3-ball with $T \cap W$ a ∂ -parallel arc. In this case, if $T \cap W$ is disjoint from H , then H is a Heegaard surface for the solid torus obtained by drilling out the arc $T \cap W$. Since H is not stabilized, it must then be a torus. In particular, since W is a 3-ball, this implies that H is meridionally stabilized, a contradiction. Thus, in particular, if $T \cap W$ has a component with both endpoints on F , then H intersects each component of $T \cap W$. We now perform a trick to guarantee that this is also the case when $T \cap W$ does have a component with endpoints on F .

Let $(\overline{W}, \overline{T})$ be the result of removing from $(W, T \cap W)$ an open regular neighborhood of all edges of $T \cap W$ which are disjoint from H . (By our previous remark, all such edges have both endpoints on $\partial W \setminus F$.) Then \overline{T} is a 1-manifold properly embedded in \overline{W} with no edge disjoint from H .

If \overline{T} has at least one edge, then by [18, Theorem 3.5] (which is a strengthening of [17, Theorem 3.1]) one of the following occurs:

- (i) $H \in vp\mathbb{H}(\overline{W}, \overline{T})$ is stabilized, boundary-stabilized along $\partial\overline{W} \setminus F$, perturbed or removable.
- (ii) H is parallel to F by an isotopy transverse to T .

Consider possibility (i). If $H \in vp\mathbb{H}(\overline{W}, \overline{T})$ is stabilized, boundary-stabilized along $\partial\overline{W}$, or removable then $H \in vp\mathbb{H}(W, T)$ would have a generalized stabilization or be removable with removing discs disjoint from the vertices of T , an impossibility. If $H \in vp\mathbb{H}(\overline{W}, \overline{T})$ is perturbed, $H \in vp\mathbb{H}(W, T)$ has a perturbation (since \overline{T} is a 1-manifold), also an impossibility. Thus, (i) does not occur. Possibility (ii) does not occur since, none of the v.p.-compressionbodies of $(M, T) \setminus \mathcal{H}$ are trivial product compressionbodies adjacent to \mathcal{H}^- .

We may assume, therefore, that \overline{T} has no edges. Then by [15], H is either parallel to F by an isotopy transverse to T or is boundary-stabilized along $\partial\overline{W}$. The former situation contradicts the assumption that $(M, T) \setminus \mathcal{H}$ contains no trivial product compressionbody adjacent to $F \subset \mathcal{H}^-$. In the latter situation, since H is boundary-stabilized in \overline{W} , then it has a generalized stabilization as

a surface in $\overrightarrow{vp\mathbb{H}}(W, T)$, contradicting the assumption that \mathcal{H} is reduced. Thus, once again, F is not boundary-parallel in $M \setminus T$. We have shown, therefore, that no component of \mathcal{H}^- is ∂ -parallel and, thus, that each component of \mathcal{H}^- is c-essential in (M, T) .

It remains to show that if (M, T) is irreducible and if some component of \mathcal{H} is a sphere disjoint from T , then $T = \emptyset$ and $M = B^3$ or $M = S^3$. Assume that (M, T) is irreducible. We have already remarked that since each component of \mathcal{H}^- is c-essential no component of \mathcal{H}^- is a sphere disjoint from T . We now show that no component H of \mathcal{H}^+ is a sphere disjoint from T , unless $T = \emptyset$ and M is S^3 or B^3 . Suppose that there is such a component $H \subset \mathcal{H}^+$. Let (C, T_C) and (D, T_D) be the v.p.-compressibodies on either side of H . By the definition of v.p.-compressionbody the surfaces $\partial_- C$ and $\partial_- D$ are the unions of spheres and T_C and T_D are the unions of ghost arcs. Consider the graphs $\Gamma_C = \partial_- C \cup T_C$ and $\Gamma_D = \partial_- D \cup T_D$ (thinking of the components of $\partial_- C$ and $\partial_- D$ as vertices of the graph.) By the definition of v.p.-compressionbody since H is a sphere disjoint from T , the graphs Γ_C and Γ_D are the union of trees. If either Γ_C or Γ_D has an edge, then a leaf of Γ_C or Γ_D is a sphere intersecting T exactly once. This contradicts the irreducibility of (M, T) . Consequently, both T_C and T_D are empty. Since no spherical component of \mathcal{H}^- is disjoint from T , this implies that $\partial_- C \cup \partial_- D$ is a subset of ∂M . Since $M \setminus T$ is irreducible, this implies that $\partial_- C \cup \partial_- D$ is either empty or a single sphere. Consequently, M is either S^3 or B^3 and $T = \emptyset$. \square

9. DECOMPOSING SPHERES

The goal of this section is to show that if we have a bridge surface for a composite knot or graph, we can untelescope it so that a summing sphere shows up as a thin level. We also show that cutting open along a thin surface creates induced oriented multiple bridge surfaces.

We start with a simple observation (likely well-known) that compressing essential twice and thrice-punctured spheres results in a component which is still essential. The proof is straightforward so we leave it to the reader.

Lemma 9.1. *Assume that (M, T) is a 3-manifold graph pair with no component of ∂M a sphere intersecting T exactly once. Suppose that $P \subset (M, T)$ is an essential sphere with $|P \cap T| \leq 3$. Let P' be the result of compressing P along an sc-disc D . Then at least one component of $P' \subset (M, T)$ is an essential sphere intersecting T at most 3 times.*

Theorem 9.2. *Suppose that (M, T) is a (3-manifold, graph) pair such that no component of ∂M is a sphere intersecting T exactly once. Suppose that there is an essential sphere $P \subset (M, T)$ such that $|P \cap T| \leq 3$. If $\mathcal{H} \in \overrightarrow{vp\mathbb{H}}(M, T)$ is locally thin, then some component F of \mathcal{H}^- is an essential sphere with $|F \cap T| \leq 3$. Furthermore, if $M \setminus T$ is irreducible or if T is irreducible, then $|F \cap T| \leq |P \cap T|$ and F can be obtained from P by a sequence of compressions using sc-discs.*

Proof. We begin by recording how an sc-compression affects the number of punctures. Observe that any sphere in (M, T) intersecting T exactly once is essential. Let $P \subset (M, T)$ be any essential sphere with $|P \cap T| \leq 3$. If $P \cap T = \emptyset$, then an sc-compression creates two components, each intersecting T at most once. If $|P \cap T| = 1$, then an sc-compression creates two components, one of which is still a sphere intersecting T once, and is therefore essential. If $|P \cap T| = 2$, then the sc-compression creates two components P_1 and P_2 such that $\{|P_1 \cap T|, |P_2 \cap T|\}$ is one of $\{0, 2\}$, $\{1, 1\}$, $\{1, 3\}$, $\{2, 2\}$. Since any sphere intersecting T exactly once is essential, the sphere produced by the theorem does not intersect T any more times than does the original P . If $|P \cap T| = 3$, and if P_1 and P_2 are the components, then $\{|P_1 \cap T|, |P_2 \cap T|\}$ is one of $\{0, 3\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 3\}$. Thus, as before, the sphere created by the theorem does not intersect T any more times than does the original. Consequently, in combination with Lemma 9.1, if either $M \setminus T$ is irreducible or if T

is irreducible, then any sc-compression of an essential sphere $P \subset (M, T)$ with $|P \cap T| \leq 3$ results in a component which is an essential sphere intersecting T no more times than the original sphere.

Without loss of generality, we may assume that the given P was chosen so that no sequence of isotopies and sc-compressions reduces $|P \cap \mathcal{H}^-|$. The intersection $P \cap \mathcal{H}^-$ consists of a (possibly empty) collection of circles. We show it is, in fact, empty. Suppose, for a contradiction, that γ is a component of $|P \cap \mathcal{H}^-|$. Without loss of generality, we may suppose it is innermost on P . Let $D \subset P$ be the unpunctured disc or once-punctured disc which it bounds. Since \mathcal{H}^- is c-incompressible, γ must bound a zero or once-punctured disc E in \mathcal{H}^- . Thus, if $|P \cap \mathcal{H}^-| \neq \emptyset$, then there is a component of the intersection which is inessential in \mathcal{H}^- .

Let $\zeta \subset P \cap \mathcal{H}^-$ be a component which is inessential in \mathcal{H}^- and which, out of all such curves, is innermost in \mathcal{H}^- . Let $E \subset \mathcal{H}^-$ be the unpunctured or once-punctured disc it bounds. Observe that ζ also bounds a zero or once-punctured disc on P . If E is not an sc-disc for P , then we can isotope P to reduce $|P \cap \mathcal{H}^-|$, contradicting our choice of P . Thus, E is an sc-disc. By Lemma 9.1, compressing P along E creates two spheres, at least one of which intersects T no more than 3 times and is essential in the exterior of T . Since this component intersects \mathcal{H}^- fewer times than does P , we have contradicted our choice of P . Hence $P \cap \mathcal{H}^- = \emptyset$.

We now consider intersections between P and \mathcal{H}^+ . If some component of \mathcal{H}^- is a once-punctured sphere, we are done, so assume that no component of \mathcal{H}^- is a once-punctured sphere. By cutting open along \mathcal{H}^- and replacing (M, T) with the component containing P , we may assume that $\mathcal{H}^- = \emptyset$ and that $H = \mathcal{H}$ is connected. Apply Theorem 8.2 to P (in place of F) to see that we can isotope H transversally to T in $M \setminus \mathcal{H}^-$ so that one of the following occurs:

- (1) $H \cap P = \emptyset$.
- (2) $H \cap P$ is a non-empty collection of curves, each of which is essential in P .
- (3) H is sc-weakly reducible.

Since \mathcal{H} is locally thin in $\overrightarrow{vp}\mathbb{H}(M, T)$, (3) does not occur. Since P contains no essential curves, (2) does not occur. Thus, $H \cap P = \emptyset$.

Let (C, T_C) be the component of $M \setminus \mathcal{H}$ containing P . By Lemma 3.3, after some sc-compressions, P is parallel to a component of $\mathcal{H}^- \cup \partial M$. By Lemma 9.1, P is parallel to a component of \mathcal{H}^- and we are done. \square

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